Effort-Maximizing Contingent Prize Allocation Rules in Sequential Three-Battle Contests

Xin Feng† Jingfeng Lu‡

May, 2015

Abstract

This paper studies the effort-maximizing prize allocation rules in sequentially played three-battle contests. The organizer has a fixed prize budget, and rewards the players contingent on the number of battles they win. The battles are played between two opposing players or between selected pairs of players from two opposing teams. A full spectrum of contest technologies in the Tullock family is accommodated. We find a winner-take-all best-of-three contest is optimal for team competitions. For competitions between two individuals, the optimal design varies with the contest technology: when the discriminatory power is within the low range, a winner-take-all best-of-three contest remains optimal; when the discriminatory power falls in the intermediate and high ranges, the optimal design takes a form of best-of-three contest with both a contest prize to the grand winner of the whole contest and uniform battle prizes to battle winners. For intermediate range, the battle prize increases with the discriminatory power but never goes beyond one-third of the total prize. For the high range, interestingly, a wide span of battle prizes ranging from zero to one-third of the total prize is optimal. Therefore, in general additional award should be allocated to the grand winner of the whole contest. Our findings thus rationalize the commonly observed winner-take-all prize structure as well as intermediate prizes in sequential multi-battle contests.

JEL Classification Numbers: D72, D74, D81

Keywords: Colonel Blotto Game, Contingent Prize Allocation, Effort Maximization, Multi-Battle Contest, Team Contest with Multiple Battles.

1 Introduction

Dynamic multi-battle contests are abundant in reality. Many economic and social competitions, including patent races, electoral campaigns and sports, can be viewed as contests in which opposing parties expend non-refundable costly effort to compete in multiple battles. A unique feature naturally arises

---

*We thank Pitchik Carolyn, Roy Chen, Yi-Chun Chen, Jeff Ely, Qiang Fu, Samuel Häfner, Chiu-Yu Ko, Haoming Liu, Martin Osborne, Mikhail Safronov, Wing Sun, Cédric Wasser, Songfa Zhong for helpful comments and suggestions. All remaining errors are our own.

†Xin Feng: Department of Economics, National University of Singapore, Singapore 117570. a0078175@u.nus.edu.

‡Jingfeng Lu: Department of Economics, National University of Singapore, Singapore 117570. ecsljf@nus.edu.sg.

1Please refer to Konrad and Kovenock (2009), Kovenock and Roberson (2012) among many others for examples on multi-battle contests.
from the multi-round nature of such competitions: The final winner of the whole contest as well as contestants’ rewards must in general be determined by the outcomes of all battles but not that of a single battle. For example, in a widely adopted winner-take-all best-of-\((2n + 1)\) contest, a party wins the contest if and only if the party wins the majority of the \(2n + 1\) battles, and the winner takes away all the prize.

There is a wide range of diversities in the observed prize structures in multi-battle contests. It is often the case that the final winner is determined according to the above mentioned “majority” rule and the prize is split between the winner and loser by fixed shares, which are not related to the numbers of battles won by the players. In the finals of the 2013 US Open tennis tournament, the winners receive a prize of $2.6 million, while the losers receive $1.3 million. In the finals of the 2015 OUE Singapore Open, the winners receive about $23,000 and the loser receives $11,400. The prizes for the runner-ups are awards for getting into the finals. Therefore, for the finals, the effective prize structure is literally a winner-take-all to the grand winners. On the other hand, in many occasions, intermediate prizes are awarded to winners of component battles. For instance, in Formula I car races, each Grand Prix generates some benefits to the winner, besides the grand championship that is awarded on an annual basis. Similarly, the PGA tour awards winners of all component tournaments, and a grand prize is awarded to the overall best performer at the end of the tour. These practices exemplifies how prize allocations can be contingent on the players’ numbers of winning battles. In general, the prize structures could depend on both the battle outcomes and contest outcome, but are not necessarily restricted to contest prizes and battle prizes.²

Interesting questions thus arise: How do the contest organizers’ choices of the prize allocation rules depend on the contest structures? In particular, for which situations should the allocation rule solely rely on the performance aggregated over all the battles, i.e. the final winning status of the whole contest, and for which situations should the players be awarded separately in each individual battle but not on their aggregate performance? Why the commonly observed prize structures usually take a simple form of a combination of a grand contest prize and component battle prizes, while the contest organizers have access to more sophisticate contingent prize allocation rules?

In this paper, we aim to rationalize commonly adopted prize allocation rules from the perspective of effort elicitation by a contest organizer who can flexibly reward the contestants based on their numbers of winning battles. For this purpose, we adopt environments of dynamic three-battle contests that allow for a whole spectrum of contest technology in the Tullock family, and fully characterize the effort-maximizing prize structures among all feasible allocation rules that are contingent on battle outcomes. Sequential battles can be played either between two fixed players or by varying pairs of players from two opposing teams. We find that the optimal design takes a form of best-of-three contests with both a contest prize to the grand winner of the whole contest and uniform battle prizes to battle winners. The player structure and the discriminatory power of the contest technology play crucial roles in determining the optimal shares of these prizes.

²The contest outcome, however, is eventually determined by the battle outcomes. In this sense, there is no loss of generality to describe all feasible prize structures as prize allocation rules that are solely contingent on the battle outcomes.
We first study the optimal contingent prize allocation rule that elicits the maximum aggregate effort in a sequential-play multi-battle contest between two risk neutral players with unit marginal effort cost. In every component battle, both players observe the outcomes of previous battles and exert effort simultaneously. We allow a full spectrum of contest technology within the Tullock family in component battles, which are indexed by the discriminatory power \( r \) of the corresponding success function. Specifically, the contest organizer has a fixed budget (normalized as 1) which can be used to fund nonnegative prizes to competing parties in a three-battle contest.\(^3\) She has the flexibility of fully allocating this budget contingent on the outcomes of the battles, i.e. the sum of wins of each party, subject to a monotonicity condition which requires a party with more wins being rewarded a weakly higher prize.\(^4\) A particularly interesting but rather intricate issue is whether and how a positive prize should be granted to a player with a single win, as will be illustrated in our analysis.

We fully characterize the optimal contingent prize allocation for every discriminatory power \( r(>0) \). For each \( r \), we first characterize the subgame perfect equilibrium by backward induction for each eligible contingent prize allocation rule. We then compare across all eligible contingent prize allocation rules to identify the optimal rule. The procedure requires lengthy computations and multi-step comparison. In particular, computing players’ total expected effort for a given prize structure requires aggregating the players’ efforts across every possible path. Because of the difficulties generated by potential ex-post asymmetry due to the sequential nature of the contest and its effect on players’ strategies, we have to consider separately multiple overlapping subsets of eligible prize structures, and obtain the optimal prize allocation rule within each subset. A comparison across all these restricted optimums yield the globally optimal prize allocation rule.

We find that the optimal prize allocation rule crucially depends on the discriminatory power \( r \) of the contest technology adopted in component battles. The discriminatory power in a Tullock contest measures the importance of a player’s effort in determining his winning probability. A higher discriminatory power \( r \) means that the winning chances are more determined by players’ effort rather than other random factors that also affect the contestants’ performances.\(^5\) Specifically, when the discriminatory power is low, a simple winner-take-all best-of-three contest is optimal; when the discriminatory power falls in the intermediate and high ranges, the optimal design takes a form of best-of-three contest with both a contest prize to the grand winner of the whole contest and uniform battle prizes to battle winners. For the intermediate range, the battle prize increases with the discriminatory power but never goes beyond one-third of the total prize. For the high range, interestingly, a wide span of battle prizes ranging from zero to one-third of the total prize is optimal.

The economics and intuitions behind these characterizations can be illustrated as follows. For convenience, we use \( v(n) \) to denote the prize awarded to a player winning \( n \in \{0, 1, 2, 3\} \) battles. It is natural that \( v(0) = 0 \) (and thus \( v(3) = 1 \)) is necessary to elicit maximal effort from players as

\(^3\)If the organizer’s budget is indivisible, she can equivalently use winning probabilities as design instruments. For convenience, we assume all prize budget must be exhausted.

\(^4\)We assume the prize allocation does not depend on the identities of the competing parties.

\(^5\)Fu and Lu (2012a) provide a microfoundation for nested Tullock contests from a noisy ranking perspective.
rewarding a player without a single win definitely dampens the players’ incentive. The more interesting and intricate trade-off lies in the balance between \( v(1) \) and \( v(2) \), the prizes for a single win and two wins. In other words, should a positive prize be granted to a player with a single win? If yes, what is the optimal level for it? How should it depend on the discriminatory power \( r \)?

We first introduce a useful fact that dramatically facilitates illustrating the impact of a marginal change in \( v(1) \): Any eligible prize structure \( \{v(n), n = 0, 1, 2, 3\} \) with \( v(0) = 0 \) and \( v(3) = 1 \) is equivalent to a combination of a grand contest prize \( v_G \) to the grand winner who wins at least two battles and a uniform battle prize \( v_B \) to the winner of each battle.\(^6\) In particular, we have \( v_B = v(1) \) and \( v_G = 1 - 3v(1) \). Therefore, the trade-off between \( v(1) \) and \( v(2) \) reduces to the trade-off between battle prizes and contest prize. A \( \Delta(>0) \) increase in battle prize \( v_B \) means a three-time drop in contest prize \( v_G \). These changes in the contest prize and battle prize have opposite impacts on the total effort induced. While it is clear that the decrease in contest prize would lower the total effort supply, the increase in battle prizes would enhance effort supply through multiple channels. First, the increase in battle prize directly leads to higher effective prize spreads in component battles, which contribute to higher effort supply in each battle. Second, a higher battle prize mitigates the well established discouragement effect in the second battle, which increases the chance that the final winner is determined only after the third battle is fought. The mitigated discouragement effect thus tends to enhance the incentive provided by the contest prize. More importantly, this mitigation effect is stronger for higher discriminatory power \( r \), i.e. when effort is more effective in determining the winner.

Based on the above discussions, when discriminatory power \( r \) is low, the positive effect from a higher battle prize tends to be small. In this case, the negative effect of a lower contest prize tends to dominate, which leads to the optimality of zero battle prize. As the discriminatory power \( r \) moves into the intermediate range, the mitigation in the discouragement effect gets stronger, which renders the optimality of a positive battle prize that increases with \( r \). When \( r \) moves into a higher range, i.e. \( r \geq 2 \), the rents are fully dissipated in the first battle in which the two players’ prize spreads are symmetric. As a result, any eligible prize structure renders the same level of total effort supply.

For comparison purpose and to strengthen the point illustrated above, we further analyze sequential-play three-battle team contests. The multi-battle team contests have been analyzed by Fu, Lu and Pan (2015). In their model, two teams with an equal number of players compete in a contest. Players from rival teams form pairwise matches to fight in multiple component battles sequentially. A team wins if and only if its players secure a majority number of victories. Each player benefits from his team’s win, while he can also receive a private reward for winning his own battle. They find that the strategic momentum effect typically identified in dynamic multi-battle contests with two players is completely nullified in this team contest setting. Häfner (2012) studies multi-battle team contests in a tug-of-war setting with sequential battles and an all-pay auction technology in each battle. In our model, the prize awarded to a team is a public good to every member within the team and the organizer endogenously choose the optimal contingent rule to award the two teams based their number of winning

\(^6\)This fact is easy to establish and is further elaborated in Section 3.5.
battles. We find a winner-take-all best-of-three contest is optimal in this environment for any positive discriminatory power \( r \). The intuition behind the result is rather clear. Even when there is no battle prize, the discouragement effect does not exist in team contests. Thus in team contests the channel no longer exists for a positive battle prize to boost effort supply by mitigating the discouragement effect. As a result, zero battle prize turns out to be optimal. The optimality of zero battle prize in team contests further confirms that the optimality of a positive battle prize in contests between same two individual players is mainly due to the mitigation of the discouragement effect.

Our paper primarily belongs to the well established literature on multi-battle contests. Environments where the battles are contested sequentially have been analyzed by Harris and Vickers (1987), Ferall and Smith (1999), Klumpp and Polborn (2006), Konrad and Kovenock (2009), McFall, Knoeber and Thurman (2010), Malueg and Yates (2010) and Sela (2011) among others. Harris and Vickers (1987) study a multi-battle patent race. Klumpp and Polborn (2006) model U.S. presidential primaries as a multi-battle dynamic contest between two candidates. Malueg and Yates (2010) study the players’ strategic effort supply in best-of-three contests and test the theoretical prediction empirically using tennis data. All these studies identify the so-called strategic momentum effect in dynamic multi-battle contests with two players. Konrad and Kovenock (2009) provide a complete characterization of the unique subgame perfect equilibrium in multi-battle contests with intermediate prizes, in which the component contests are modelled as all-pay auctions. They find that even a large lead by one player might not fully discourage the other when a component battle awards a positive intermediate prize.\(^7\) Sela (2011) compares the best-of-three all-pay auction to the standard one-stage all-pay auction.

Our paper further this line of research by studying how intermediate prizes can be optimally designed and utilized to mitigate the strategic momentum effect and provide the best incentive to the contestants. To our best knowledge, our paper is the first study on optimal contingent prize allocation in the environment of dynamic multi-battle contests. A winner-take-all prize structure together with a “majority” winning rule is commonly adopted in practice in multi-battle contests, and typically assumed in the literature. Our paper rationalizes the optimality of this popular prize allocation rule in a wide range of environments in dynamic multi-battle contest context. This finding extends the validity of the winner-take-all principle which has been established in many other settings, including Clark and Riis (1998), Krishna and Morgan (1998), Moldovanu and Sela (2001), Lai and Matros (2005), Fu and Lu (2012b) and Schweinzer, Paul and Segev (2012) among others.

We find that a prize allocation rule that rewards every positive number of wins (i.e. split awards contingent on overall battle outcomes) generates the maximal expected total effort when the battles proceed sequentially between the same two players and the discriminatory power \( r \) is sufficiently high. In particular, the reward for winning one battle increases and the reward for winning two battles decreases with \( r \) is in a middle range. These findings suggest intermediate prizes can facilitate effort elicitation at many occasions in dynamic multi-battle contests between two players.\(^8\)

---

\(^7\)Irfanoglu et al. (2010) and Mago and Sheremeta (2012) test these theoretical implications by experiments.

\(^8\)Mago, Sheremeta and Yates (2013) find that rewarding intermediate battle prizes could boost effort supply in dynamic contests even with small \( r \). However, in their study, the extra battle prizes are funded by additional budget.
The rest of the paper proceeds as follows. In Section 2, we set up the model in the framework of sequentially played three-battle contests between two same players, and introduce some notations and useful existing results in two-player single stage simultaneous-play contests of Tullock family allowing for the full span of discriminatory powers. In Section 3, we derive the subgame perfect equilibrium for every eligible contest technology by backward induction, and identify the optimal contingent prize allocation rule. In Section 4, we study the optimal contingent prize allocation rule in the environment of sequential team contests with multiple pairwise battles. Section 5 provides some concluding remarks. The appendix collects some technical proofs.

2 The model setup

Two players \(A\) and \(B\) compete in a dynamic three-battle contest. Both of them are risk neutral and have unit marginal effort cost. They fight the three battles sequentially, and observe the past outcome (i.e. the state of the contest) before exerting effort in the current battle.

The contest organizer has a prize budget \(V\), which is normalized to 1. The organizer’s prize allocation rule is contingent on the contest outcome, i.e. the number of battles the each player wins. Let \(v(n)\), \(n \in \{0, 1, 2, 3\}\) denote the prize that a player wins if he wins \(n\) battles. Alternatively, \(v(n)\) can be interpreted as the winning probability of a single indivisible grand prize with value 1. We thus have the following feasibility restrictions on the prize allocations:

\[
v(n) + v(n') = 1, \forall n, n' \in \{0, 1, 2, 3\}, n + n' = 3;
\]

\[
v(n) \geq v(n'), \forall n \geq n';
\]

\[
v(n) \geq 0, \forall n.
\]

The first constraint says that the sum of prizes to the two players cannot go beyond the total prize budget, in particular, it exhausts the whole budget. The second constraint says that the player with higher number of wins is awarded a higher prize. The third constraint means the prizes cannot be negative, which is natural when \(v(n)\) is interpreted as winning probabilities or the players are subject to limited liability. Note that the winning probability interpretation is particularly relevant when the prize \(V\) is indivisible.

A generalized Tullock contest technology is adopted for each component battle, in which both players exert their effort simultaneously. Let \(e_A\) and \(e_B\) denote the players’ effort in a battle. The player \(i\)’s probability of winning the battle is specified by \(p_i = \frac{e_i}{e_i + e_j}\) where \(i, j \in \{A, B\}\) and \(r \in (0, \infty)\) denotes the discriminatory power of the Tullock contest.

In this paper, we study the optimal prize allocation rule that elicits the highest expected total effort in the contest.
2.1 Preliminaries: equilibrium strategies in Tullock contests

We first present some existing results on equilibrium analysis in a two-player Tullock contest with asymmetric values and an arbitrary discriminatory power \( r \). These results pave the foundation of our analysis. Consider two players \( i \) and \( j \) competing in a generalized Tullock contest with discriminatory power \( r \). The value of player \( i \) is \( v_i \); and the value of player \( j \) is \( v_j \). Without loss of generality assuming \( v_i \geq v_j > 0 \).\(^9\) Player \( i \)'s winning probability is given by 
\[
p_i = \frac{x_i^r}{x_i^r + x_j^r}
\]
where \( x_i \) and \( x_j \) denote players' efforts, and \( r \in (0, \infty) \) denotes the discriminatory power of the contest. We let \( x_i(v_i, v_j; r) \) and \( x_j(v_i, v_j; r) \) denote the players' equilibrium strategy, which can be either pure or mixed.

**Definition 1** Assume \( 0 < z \leq 1 \). A cutoff \( \hat{r}(z) \in (1, 2] \) is defined as the unique solution to
\[
r = 1 + z^r.
\]

Nti (1999) establishes that a pure-strategy equilibrium exists if and only if \( r \) is bounded from above by a cutoff \( \hat{r}(\frac{v_i}{v_j}) \leq 2 \) and provides a complete characterization of the equilibrium strategy. Wang (2010) analyzes the case of \( r \in (\hat{r}(\frac{v_i}{v_j}), 2] \) and obtains a closed-form solution to the equilibrium strategy.\(^10\) The mixed-strategy equilibrium in an all-pay auction has been analyzed extensively in the literature. Alcalde and Dahm (2010) analyze the case of \( r > 2 \). They identify an “all-pay-auction” equilibrium in mixed strategies, although a closed form solution of the equilibrium strategies remains less than explicit. These characterizations are summarized as follows:

**Lemma 1** Assuming \( v_i \geq v_j > 0 \). The equilibrium bidding strategies \( x_i(v_i, v_j; r) \) and \( x_j(v_i, v_j; r) \) are:

(i) If \( r \leq \hat{r}(\frac{v_i}{v_j}) \),
\[
  x_i(v_i, v_j; r) = \frac{r v_i^{r+1} v_j^r}{(v_i^r + v_j^r)^2},
\]
\[
  x_j(v_i, v_j; r) = \frac{r v_j^{r+1} v_i^r}{(v_i^r + v_j^r)^2}.
\]

(ii) If \( r \in (\hat{r}(\frac{v_i}{v_j}), 2] \),
\[
  x_i(v_i, v_j; r) = \left(\frac{1}{r - 1}\right)^{\frac{r}{r+1}} (1 - \frac{1}{r}) v_j,
\]
\[
  x_j(v_i, v_j; r) = \begin{cases} 
    (1 - \frac{1}{r}) v_j, & \text{with probability } q = \frac{v_j}{v_i} (\frac{1}{r - 1})^{\frac{r}{r+1}}, \\
    0, & \text{with probability } 1 - q.
  \end{cases}
\]

\(^9\) Otherwise, we can relabel the two players.

\(^10\) The cutoff \( \hat{r}(\frac{z}{r}) \) converges to 2 when \( r \) approaches \( y \), i.e., when the two players are symmetric. In that case, the particular case analyzed by Wang (2010) vanishes.
(iii) If $r > 2$,

$$
x_i(v_i, v_j; r) = \mu^*,
$$

$$
x_j(v_i, v_j; r) = \begin{cases} 
\mu^*, & \text{with probability } q = \frac{v_j}{v_i}, \\
0, & \text{with probability } 1 - q,
\end{cases}
$$

where $\mu^*$ is the (symmetric) equilibrium mixed strategy identified by Baye, Kovenock, and de Vries (1994) in a two-player Tullock contest with $r > 2$, and fully dissipates the rent in the symmetric game when both valuations equal $v_j$.

In Case (i), the proposed pure-strategy equilibrium is unique. The uniqueness of the equilibrium in Case (ii) and Case (iii), however, has not been established in the literature. In all three cases, $x_i(v_i, v_j; r)$ is a more aggressive strategy than $x_j(v_i, v_j; r)$ as long as $v_i \geq v_j$, i.e., contestant with the higher valuation tends to exert the higher effort.

In order to pin down the optimal prize structure $\{v(0), v(1), v(2), v(3)\}$ that elicits the maximum expected aggregate effort, we have to evaluate players’ incentives and calculate players’ equilibrium efforts respectively according to the above equilibrium bidding strategies for each possible history. Note that the equilibrium bidding strategies above depend on $\tilde{r}(\frac{v_j}{v_i})$ and $\tilde{r}(\frac{v_i}{v_j})$ varies with the ratio of the two players’ valuations. And it is the prize structure that determines the two players’ incentives and thus player’s incentives in each battle. Consequently, $\tilde{r}(\frac{v_j}{v_i})$ also varies with the prize structure $\{v(0), v(1), v(2), v(3)\}$.

### 2.2 Other notations

To derive the optimal prize allocation rule for a fixed $r$ associated with the Tullock technology, we have to compare all feasible prize structures in terms of the induced aggregate effort. For that, we solve each involved stage contest backwards by adopting the equilibrium strategies given by Lemma 1. According to the range of discriminatory power $r$, there are three cases to discuss in Lemma 1. When we restrict to $r \in (0, 2]$, divided by a cutoff depends on two players’ incentives, two cases remain to be discussed. And this aggravates calculation difficulties for a stage contest where players’ incentives differ and vary with the feasible prize structures.

To make it clear, we analyse the whole contest under a feasible prize structure $\{v(0), v(1), v(2), v(3)\}$ for a fixed $r \in (0, 2]$. Recall the budget constraints (1), it is thus sufficient for us to focus on $(v(0), v(2))$ to fully describe the prize structure. We start from the third stage and look at players’ incentives at each reachable state. When a player has won the first two battles, this player responds to his effective prize spread $v(3) - v(2)$ in battle 3 and his opponent responds to the prize spread $v(1) - v(0)$, the two players have a common effective prize spread of winning the third battle because of the budget constraints (1). By Lemma 1(i), their common equilibrium effort supply equals $\frac{v}{2} (v(1) - v(0))$ and each of them has half chance to win the third battle. When each have won one battle, their common effective prize spread of winning the third battle is $v(2) - v(1)$. By Lemma 1(i), their common equilibrium effort supply equals $\frac{v}{2} (v(2) - v(1))$ and each wins the third battle with half probability. We proceed to the second battle,
suppose player $A$ has won the first battle, his prize spreads equal

$$w_A := [p_A(2, 0)v(3) + p_B(2, 0)v(2) - x_A(2, 0)] - [p_A(1, 1)v(2) + p_B(1, 1)v(1) - x_A(1, 1)]$$

$$= \frac{1}{2}v(3) + \frac{1}{2}v(2) - \frac{r}{4}(v(1) - v(0))] - [\frac{1}{2}v(2) + \frac{1}{2}v(1) - \frac{r}{4}(v(2) - v(1))]$$

His opponent who loses the first battle has prize spread equals

$$w_B := [p_B(1, 1)v(1) + p_A(1, 1)v(1) - x_B(1, 1)] - [p_B(2, 0)v(1) + p_A(2, 0)v(0) - x_B(2, 0)]$$

$$= [\frac{1}{2}v(2) + \frac{1}{2}v(1) - \frac{r}{4}(v(2) - v(1))] - [\frac{1}{2}v(1) - \frac{1}{2}v(0) - \frac{r}{4}(v(1) - v(0))]$$

Two players’ incentives for competing second battle become asymmetric unless the prize allocation does not reward additional prize to the player with majority wins(i.e., $v(2) - v(1) = v(1) - v(0)$). Specifically, $w_A$ reflects the incentive in the second battle of the winner of the first battle, and $w_B$ reflects the incentive in the second battle of the loser of the first battle. We define them in the following for deriving the optimal prize allocation rule when $r \in (0, 2]$.

**Definition 2** Let $w_A = (\frac{1}{2} - \frac{r}{4})[v(1) - v(0)] + (\frac{1}{2} + \frac{r}{4})[v(2) - v(1)]$, $w_B = (\frac{1}{2} + \frac{r}{4})[v(1) - v(0)] + (\frac{1}{2} - \frac{r}{4})[v(2) - v(1)]$ and $\eta = \frac{w_B}{w_A}$.

Using the budget constraints (1), $w_A$ and $w_B$ can be alternatively written as $w_A = (\frac{1}{2} - \frac{r}{4})[1 - v(2) - v(0)] + (\frac{1}{2} + \frac{r}{4})[2v(2) - 1]$ and $w_B = (\frac{1}{2} + \frac{r}{4})[1 - v(2) - v(0)] + (\frac{1}{2} - \frac{r}{4})[2v(2) - 1]$. Thus $w_A$, $w_B$ and $\eta$ only depend on $r$, $v(0)$ and $v(2)$.

By Lemma 1, when $r \in (0, 2]$, for solving the players’ equilibrium effort supply in battle 2, we have to compare the cutoff $\hat{r}(\frac{w_B}{w_A})$ with the given $r$. Note that the cutoff depends on players’ incentives to win the second battle, which can be asymmetric and vary with the prize structure $\{v(0), v(1), v(2), v(3)\}$. In particular, it is the prize structure that determines who has higher incentive to win battle 2. Consequently, for the given $r \in (0, 2]$, we may have to adopt different equilibrium strategies to solve the second stage contest as the prize structures vary. To identify the form of the two players’ equilibrium strategies in the battle 2, we introduce the overlapping sets whose union is the collection of all feasible prize allocation rules. Moreover, in each set $\mathcal{V}_i$, we fix the form of the two players’ equilibrium strategies adopted in the second battle.

We first define the whole feasible prize structures $\mathcal{V}$ in definition 3(i) and divide $\mathcal{V}$ into overlapping sets.

**Definition 3** (i) $\forall r$, we define $\mathcal{V} = \{(v(0), v(2)) : 0 \leq v(0) \leq \frac{1}{2} \leq v(2) \leq 1$ and $v(2) + v(0) \leq 1\}$; 
(ii) $\forall r \in (0, 2]$, $\mathcal{V} = \bigcup_{i=0}^{2} \mathcal{V}_i$, where we define $\mathcal{V}_0 = \mathcal{V} \cap \{(v(0), v(2)) : r \leq 1 + (\frac{w_B}{w_A})^r \text{ and } w_A \leq w_B\}$, $\mathcal{V}_1 = \mathcal{V} \cap \{(v(0), v(2)) : r \leq 1 + (\frac{w_B}{w_A})^r \text{ and } w_A \geq w_B\}$, $\mathcal{V}_2 = \mathcal{V} \cap \{(v(0), v(2)) : 1 + (\frac{w_B}{w_A})^r < r \leq 2 \text{ and } w_A \geq w_B\}$ and $\mathcal{V}_3 = \mathcal{V} \cap \{(v(0), v(2)) : 1 + (\frac{w_B}{w_A})^r < r \leq 2 \text{ and } w_A \leq w_B\}$;
(iii) \( \forall r > 2, \mathcal{V} = \bigcup_{i=0}^{r-4} \mathcal{V}_i \), where we define \( \mathcal{V}_4 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \geq v(1) - v(0)\} \) and \( \mathcal{V}_5 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \leq v(1) - v(0)\} \).

The restrictions \( 0 \leq v(0) \leq \frac{1}{2} \leq v(2) \leq 1 \) and \( v(2) + v(0) \leq 1 \) in the definition of \( \mathcal{V} \) are equivalent to the monotonicity (2) and non-negativity (3) and of \( v(n) \) under budget constraints (1).

Note that \( \mathcal{V}_2 \cup \mathcal{V}_3 \) can be empty for some \( r \in (0, 2] \). To pin down the scope we should search for the optimal prize structure, we first find for which \( r \in (0, 2] \), we could restrict our search to \( \mathcal{V}_0 \) and \( \mathcal{V}_1 \) because \( \mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \) and for which \( r \in (0, 2] \), we have to search \( \mathcal{V}_2 \cup \mathcal{V}_3 \) as well because they are no longer empty sets. For that, we first define \( \tau \) as follow.

**Definition 4** We define \( \tau \in (1, 1.2) \) as the unique solution of \( r = 1 + \left( \frac{1 - \tau}{2 + \tau} \right)^r \).

Clearly, a solution \( \tau \) must fall in (1, 2). The uniqueness of \( \tau \) follows from the facts that when \( r \in (1, 2) \), the left hand side of the equation increases with \( r \) and the right hand side decreases with \( r \). Moreover, we have \( \tau < 1.2 \) as \( \left( \frac{1 - \tau}{2 + \tau} \right)^r < 0.2 \) when \( r = 1.2 \). One can obtain the numerical solution of \( \tau \approx 1.935 \).

This \( \tau \) tells us which pairs of overlapping sets we should investigate given the whole feasible prize structures as \( r \) varies from 0 to 2. When \( r < \tau \), we only investigate \( \mathcal{V}_0 \) and \( \mathcal{V}_1 \) in which only pure equilibrium strategy is involved. While \( r \in (\tau, 2] \), in the way of searching the optimal prize structure, we have to consider \( \mathcal{V}_0 \) and \( \mathcal{V}_1 \), as well as, \( \mathcal{V}_2 \) and \( \mathcal{V}_3 \), so that the involved equilibrium strategy can be either pure or mixed. We prove it in the following Lemma.

**Lemma 2** \( \forall r \in (0, \tau] \), we have \( \mathcal{V} = \mathcal{V}_0 \cup \mathcal{V}_1 \); \( \forall r \in (\tau, 2] \), we have \( \mathcal{V}_2 \neq \emptyset \).

**Proof.** For the first part, it suffices to show that \( w_A \) and \( w_B \) satisfy either \( r \leq 1 + \left( \frac{w_A}{w_B} \right)^r \) when \( w_A \leq w_B \) or \( r \leq 1 + \left( \frac{w_B}{w_A} \right)^r \) when \( w_A \geq w_B \) for any given \( r \in (0, \tau] \). By the definitions of \( w_A \) and \( w_B \), we can easily verify that \( \left( \frac{1 - \tau}{2 + \tau} \right)^r \leq \left( \frac{w_A}{w_B} \right)^r \) when \( w_A \leq w_B \); and \( \left( \frac{1 - \tau}{2 + \tau} \right)^r \leq \left( \frac{w_B}{w_A} \right)^r \) when \( w_A \geq w_B \) for any \( r > 0 \). By the definition of \( \tau \) and the fact \( r \in (0, \tau] \), we have \( r \leq 1 + \left( \frac{1 - \tau}{2 + \tau} \right)^r \). Therefore \( r \leq 1 + \left( \frac{1 - \tau}{2 + \tau} \right)^r \leq 1 + \left( \frac{w_B}{w_A} \right)^r \);

and \( r \leq 1 + \left( \frac{1 - \tau}{2 + \tau} \right)^r \leq 1 + \left( \frac{w_B}{w_A} \right)^r \), \( \forall r \in (0, \tau] \).

For the second part, we can verify that prize structure with \( v(0) = v(1) = 0, v(2) = v(3) = 1 \) must belong to \( \mathcal{V}_2 \), \( \forall r \in (\tau, 2] \). For this prize structure, we have \( w_A = 1 + \frac{r}{4} > w_B = \frac{1}{2} - \frac{r}{4} \), it is clear that \( 1 + \left( \frac{w_B}{w_A} \right)^r = 1 + \left( \frac{1 - \tau}{2 + \tau} \right)^r < r \) when \( r \in (\tau, 2] \). So \( \mathcal{V}_2 \neq \emptyset \). \( \blacksquare \)

### 3 Optimal prize allocation

We consider the optimal prize allocation rule for three ranges of the discriminatory power \( r \). In case 1, \( r \in (0, \tau] \), in case 2, \( r \in (\tau, 2] \), and in case 3, \( r \in (2, +\infty) \). For each case, the results of Lemma 1 can be utilized to solve for the subgame perfect equilibrium and compute the expected total effort from all three battles.
3.1 Case 1: \( r \in (0, \overline{r}] \)

By Lemma 2, we restrict to the prize structures in \( V_0 \cup V_1 \) when \( r \in (0, \overline{r}] \) and pin down the aggregate equilibrium effort elicited at the subgame perfect equilibrium. The equilibrium is characterized in the proof of Lemma 3 following standard backward induction procedure. Appropriate parts of Lemma 1 are utilized in different battles depending on the relation between the players’ prize spreads and the discriminatory power in each battle.

We first pin down the expected aggregate effort induced by a prize profile in \( V_0 \cup V_1 \) in the following lemma. For any given prize structure, computing aggregate effort requires solving the players’ choices of effort on each stage game for every possible path. Since the uniqueness of the pure equilibrium described by Lemma 1(i) is established for one-shot two-player Tullock contest by Nti (1999), the corresponding equilibrium is unique on each stage game, the usual backwards induction argument shows that the subgame perfect equilibrium of the whole contest is unique under each prize structure in \( V_0 \cup V_1 \).

**Lemma 3** \( \forall r \in (0, \overline{r}], \) the aggregate expected equilibrium effort equals

\[
TE_1 = \frac{rw_A^r w_B^r}{w_A^r + w_B^r}[1 - \frac{r}{2}]w_A + (1 + \frac{r}{2})w_B + \frac{r}{2}[v(2) - 1] + \frac{w_A^r [1 - v(2) - v(0)]}{w_A^r + w_B^r}.
\] (4)

**Proof.** By Lemma 2, we only need to consider the prize structure covered by \( V_0 \) and \( V_1 \). We solve the subgame perfect equilibrium by backward induction. The ordered pair \((n_A, n_B)\) to denote the number of battles won by the ordered pair of players \((A, B)\).

We first look at the third battle. When \((n_A, n_B) = (2, 0)\), by the budget constraints (1), the two players have a common effective prize spread of winning the third battle conditional on player \( A \)'s winning the first two battles.

\[
v_A(2, 0) = v_B(2, 0) = v(3) - v(2) = v(1) - v(0) \geq 0.
\]

By Lemma 1(i), we have the following equilibrium effort supply

\[
x_A(2, 0) = x_B(2, 0) = \frac{rv_A(2, 0)}{4} = \frac{r}{4}(v(1) - v(0)).
\] (a)

Each player’s winning probability for this battle is

\[
p_A(2, 0) = p_B(2, 0) = \frac{1}{2}.
\] (b)

The same results hold when \((n_A, n_B) = (0, 2)\).

When \((n_A, n_B) = (1, 1)\), the two players’ common effective prize spread of winning the third battle is

\[
v_A(1, 1) = v_B(1, 1) = v(2) - v(1) \geq 0.
\]
By Lemma 1(i), we have the following equilibrium effort supply

\[ x_A(1, 1) = x_B(1, 1) = \frac{rv_A(2,0)}{4} = \frac{r}{4}(v(2) - v(1)). \]  

Each player's winning probability for this battle is

\[ p_A(1, 1) = p_B(1, 1) = \frac{1}{2}. \]  

We now turn to the second battle. When \((n_A, n_B) = (1, 0)\), the effective prize spread for player A is

\[ v_A(1,0) = [p_A(2,0)v(3) + p_B(2,0)v(2) - x_A(2,0)] - [p_A(1,1)v(2) + p_B(1,1)v(1) - x_A(1,1)] \]
\[ = \left[ \frac{1}{2} v(3) + \frac{1}{2} v(2) - \frac{r}{4} (v(1) - v(0)) \right] - \left[ \frac{1}{2} v(2) + \frac{1}{2} v(1) - \frac{r}{4} (v(2) - v(1)) \right] \]
\[ = w_A. \]

Analogously, the effective prize spread for player B is \(v_B(1,0) = w_B\).\(^{11}\)

Note that, by Lemma 2, the prize structures are restricted to \(\mathcal{V}_0 \cup \mathcal{V}_1\). Thus Lemma 1(i) applies to the second battle. We have the following equilibrium effort

\[ x_A(1, 0) = \frac{rv_A^{r+1}(1,0)v_B(1,0)}{[v_A(1,0) + v_B(1,0)]^2}; x_B(1, 0) = \frac{rv_B^{r+1}(1,0)v_A(1,0)}{[v_A(1,0) + v_B(1,0)]^2}. \]  

The players’ winning probabilities are

\[ p_A(1, 0) = \frac{x_A(1,0)}{x_A(1,0) + x_B(1,0)} = \frac{v_A(1,0)}{v_A(1,0) + v_B(1,0)}; \]
\[ p_B(1, 0) = \frac{x_B(1,0)}{x_A(1,0) + x_B(1,0)} = \frac{v_B(1,0)}{v_A(1,0) + v_B(1,0)}. \]

When \((n_A, n_B) = (0, 1)\), similarly we have

\[ x_A(0,1) = x_B(1,0) = \frac{rv_A^{r+1}(1,0)v_B(1,0)}{[v_A(1,0) + v_B(1,0)]^2}; x_B(0,1) = x_A(1,0) = \frac{rv_B^{r+1}(1,0)v_A(1,0)}{[v_A(1,0) + v_B(1,0)]^2}. \]

Then players’ winning probabilities are

\[ p_A(0, 1) = p_B(1, 0) = \frac{v_A(1,0)}{v_A(1,0) + v_B(1,0)}; p_B(0, 1) = p_A(1, 0) = \frac{v_B(1,0)}{v_A(1,0) + v_B(1,0)}. \]

Now we consider the first battle. By \((a) - (h)\), the effective prize spreads are symmetric across the

\(^{11}\)See Definition 2.
two players:

\[ v_A(0, 0) = v_B(0, 0) \]

\[ = \{p_A(1, 0)p_A(2, 0) + p_B(2, 0)v(2) - x_A(2, 0) \}
\[ + p_B(1, 0)p_A(1, 0)v(2) + p_B(1, 0)v(1) - x_A(1, 1) - x_A(1, 0) \}
\[ - \{p_A(0, 1)p_A(1, 0)v(2) + p_B(1, 0)v(1) - x_A(1, 1) \}
\[ + p_B(0, 1)p_A(0, 2)v(2) + p_B(0, 2)v(0) - x_A(0, 2) \}
\[ - x_A(0, 1) \}
\[ = \frac{rv_A^r(0,v(1),0,r_A^r(1),1,v_B^r(1),0)}{v_A^r(1,0)+v_B^r(1,0)}[v_B(1,0)-v_A(1,0)] + \frac{v_A^r(1,0)}{v_A^r(1,0)+v_B^r(1,0)}[v(2)-v(0)]. \]

Applying Lemma 1(i), we have the following equilibrium effort

\[ x_A(0,0) = x_B(0,0) = \frac{r}{4}v_A(0,0), \]
and players’ winning probabilities are

\[ p_A(0,0) = p_B(0,0) = \frac{1}{2}. \]

By (a) – (h), together with the calculation above, the aggregate effort over all the three battles equals

\[ TE_1 = 2x_A(0,0) + \{x_A(1,0)+x_B(1,0)\} + p_A(1,0)[x_A(2,0)+x_B(2,0)] + p_B(1,0)[x_A(1,1)+x_B(1,1)] \]

\[ = \frac{rw_A^rwr_B^r}{[w_A^r+w_B^r]^2}[(1-\frac{r}{2})w_A + (1+\frac{r}{2})w_B] + \frac{r}{2}(v(2)-v(1)) + r(v(1)-v(0))\frac{w_A^r}{w_A^r+w_B^r}, \]

which gives the deserved result by incorporating the budget constraints. ■

Note that \( TE_1 \) only depend on prizes \( v(0) \) and \( v(2) \), with \( 0 \leq v(0) \leq \frac{1}{2} \leq v(2) \leq 1 \) and \( v(2)+v(0) \leq 1 \). The next lemma pins down how \( TE_1 \) changes with \( v(0) \) for given \( v(2) \in \left[ \frac{1}{2}, 1 \right] \).

**Lemma 4** \( \forall r \in (0, \overline{r}], \) we have \( \frac{dTE_1}{dv(0)} < 0, \forall v(0) \in [0, 1] \). Therefore, we have \( v(0) = 0 \) and \( v(3) = 1 \) at the optimum.

**Proof.** See Appendix. ■

Lemma 4 shows that \( TE_1 \) always decreases with \( v(0) \) in the eligible range determined by \( v(2) \). We thus conclude that the optimal \( v(0) = 0 \) and \( v(3) = 1 \). This result is quite intuitive. In order to increase effort, no prize should be given to an agent who loses all three battles. Doing so would save budget for rewarding better performance and increase effective prize spreads for players.

We next pin down prizes \( v(1) \) and \( v(2) \). The following definition and properties help us to prove Theorem 1 in which we access the effect of an increase in \( v(2) \) on the aggregate effort \( TE_1 \) in this case.

**Definition 5** Define \( \eta \) to be the unique solution of \( D(\frac{1-r}{2}, r) = 0 \) on \( (0, \overline{r}] \), where \( D(\eta, r) \equiv \left((\frac{3r}{2}-1)+(\frac{3r}{2}+1)\eta\right)[(1-\frac{r}{2})(-r-1+(r-1)\eta^r) + (1+\frac{r}{2})(-r+1+(r+1)\eta^r)\eta + 2\eta(1+\eta^r)(2-\frac{3r^2}{4}+\eta^r). \)
The existence and uniqueness of \( r \) is revealed by the following property.

**Property 1** \( D(\frac{1-r}{2+r}, r) \) strictly decreases with \( r \in (0, \bar{r}] \) and has a unique root of \( r \approx 1.0884 \) in this range.

**Proof.** See Appendix. ■

**Property 2** \( \forall r \in (0, \bar{r}], D(\eta, r) \) increases with \( \eta \) when \( \eta \geq \frac{1-r}{2+r} \).

**Proof.** See Appendix. ■

We now are ready to fully pin down the unique optimal prize structure in this case.

**Theorem 1** In a sequential three battle contest with two players, then expected aggregate effort is maximized only when \((v(0), v(1), v(2), v(3))\) equals

(i) \((0, 0, 1, 1)\) if \( r \in (0, \underline{r}] \), where \( \underline{r} \in (0, \bar{r}] \) satisfies \( D(v = \frac{1-r}{2+r}, r)_{|r=\underline{r}} = 0 \).

(ii) \((0, 1 - v^*_r(2), v^*_r(2), 1)\) if \( r \in (\underline{r}, \bar{r}] \), where \( v^*_r(2) \in (\frac{5}{8}, 1) \) can be determined from \( D(\eta(v(0), v(2), r), r)|_{v(0)=0} = 0 \).

**Proof.** Based on Lemma 4, to fully identify the optimal prize allocation for a given \( r \in (0, \bar{r}] \), we need to maximize \( TE_1(v(0), v(2))|_{v(0)=0} \) by searching all feasible \( v(2) \in [\frac{1}{2}, 1] \). For this purpose, we look at the first order derivative of \( TE_1(v(0), v(2))|_{v(0)=0} \) with respect to \( v(2) \) and find that

\[
\frac{d}{dv(2)} TE_1(v(0), v(2))|_{v(0)=0} \quad \text{sign} \quad D(\eta(v(0) = 0, v(2), r), r), \quad \eta(v(0) = 0, v(2), r) := \frac{w_B}{w_A} |_{v(0)=0} = \frac{(\frac{1}{2} + \frac{r}{2})(1-v(2))+(\frac{1-r}{2})(2v(2)-1)}{(\frac{1}{2}-\frac{r}{2})(1-v(2))+(\frac{1+r}{2})(2v(2)-1)}
\]

is a function of \( v(2) \) and \( r \) since \( v(0) = 0 \) and \( \eta = \frac{w_B}{w_A} \).

Note \( \eta(v(0) = 0, v(2), r) \) decreases with \( v(2) \in [\frac{1}{2}, 1] \) when \( r \leq 2 \). Thus \( \frac{1-r}{2+r} \geq \eta(v(0) = 0, v(2), r) \geq \frac{1-r}{4} \). By Property 2 and 3, we have \( D(\eta(v(0) = 0, v(2), r), r) \geq D(\frac{1-r}{2+r}, r) > 0, \forall v(2) \in [\frac{1}{2}, 1], r \in (0, \bar{r}] \), thus an increase in \( v(2) \) will raise aggregate effort when \( r \in (0, \underline{r}] \). By the Property 2, we know \( D(\eta(v(0) = 0, v(2) = 1, r), r) = D(\frac{1-r}{2+r}, r) < 0, \forall r \in (\underline{r}, \bar{r}] \), while \( D(\eta(v(0) = 0, v(2) = \frac{3}{4}, r), r) = (3r^2 + 12) > 0 \).

By Property 3, for each \( r \in (\underline{r}, \bar{r}] \), there exists a unique solution \( v^*_r(2) \) of \( D(\eta(v(0) = 0, v(2), r), r) = 0 \) that can induce the maximum aggregate effort. For each \( r \in (\underline{r}, \bar{r}] \), we have \( v^*_r(2) \in (\frac{5}{8}, 1) \). By simulation, we can verify that \( v^*_r(2) \) decreases with \( r \in (\underline{r}, \bar{r}] \).

We now proceed to prove that the optimal prize structure is unique for each \( r \in (0, \bar{r}] \). Under each feasible structure in \( \mathcal{V}_0 \cup \mathcal{V}_1 \), the subgame perfect equilibrium of the whole contest is unique because of the uniqueness of the pure equilibrium on each stage contest for each possible history. Together with the above searching of all feasible prize structures, the uniqueness of the optimal prize structure we identify for each \( r \in (0, \bar{r}] \) follows. ■

---

\(^{12}\)Recall that Nti (1999) establishes the existence and uniqueness of a pure-strategy equilibrium in one-shot Tullock contest when \( r \) is bounded from above by a cutoff \( \hat{r}(\frac{1}{\alpha}) \leq 2 \). When \( r \in (0, \bar{r}] \), \( r \leq \hat{r}(\frac{1}{\alpha}) \) holds for each stage game under any feasible prize structure.
Theorem 1 can also be rewritten by using alternative setup as follow. In a sequential three battle contest with two players, expected aggregate effort is maximized only when \((v, V)\) equals (i) \((0, 1)\) if \(r \in (0, \overline{r}]\); (ii) \((1-v^*_r(2), 3v^*_r(2) - 2)\) if \(r \in (\overline{r}, \overline{r}]\), where \(v^*_r(2) \in \left(\frac{2}{3}, 1\right)\) can be determined from 
\[D(\eta(v(0), v(2), r), r)|_{v(0)=0} = 0\] and thus \(V = 3v^*_r(2) - 2 \geq 0\).

### 3.2 Case 2: \(r \in (\overline{r}, 2]\)

We now turn to the case where \(r \in (\overline{r}, 2]\). Recall in this case, we have we have \(V = \bigcup_{i=0}^{3} V_i\) by Property 1. We first show that no prize profiles outside \(V_1\) can be optimal. We then focus on \(V_1\) to fully characterize the optimal profile when \(r \in (\overline{r}, 2]\).

Since Lemma 5 establishes that any prize structure in \(V_0\) is dominated by one in \(V_1\), we can ignore any prize structure in \(V_0\). Note that \(V_1\) covers the prize structures such that \(w_A \geq w_B\), i.e. the winner of the first battle has a higher prize spread.

**Lemma 5** \(\forall r \in (\overline{r}, 2]\), any prize profile in \(V_0\) is strictly dominated by a prize profile in \(V_1\), unless it is the profile with \(v(0) = 0, v(2) = \frac{2}{3}\), which belongs to \(V_0 \cap V_1\).

**Proof.** For any prize profile in \(V_0\) or \(V_1\), aggregate effort is denoted by \(TE_1\) in (4). Take any prize profile in \(V_0\) and let it differ from \((v(0) = 0, v(2) = \frac{2}{3})\). We construct a prize structure in \(V_1\) that induces a strictly higher level of effort than the given prize structure in \(V_0\). The detailed construction is in the appendix. 

By using the alternative setup to explain the result, Lemma 5 says that the optimal prize structure should satisfy \(V \geq 0\). This seems natural that bonus \(V \geq 0\) should be provided for the player who wins two battles for the purpose of effort maximization.

The following lemma provides two lower bounds for the maximal effort inducible in \(V_1\). The two lower bounds are the effort levels generated by two particular prize profiles, one in \(V_1\) and one in \(V_0\).

**Lemma 6** \(\forall r \in (\overline{r}, 2]\), we have \(\max_{V_1} TE_1 \geq \max \left\{ \frac{5}{2} + \frac{(r-1)(2-r)}{1+(r-1)^2}, \frac{5r^2+5r-2+(r^2+r+1)(r-1)^{\frac{3}{2}}}{(2r^2-1)(r-1)^2 + \frac{4r}{5} + 1} \right\} \).

**Proof.** Pick two prize profiles \((v(0) = 0, v(2) = \frac{2r[(r-1)^{\frac{3}{2}}+1]}{3r+3r(r-1)^{\frac{3}{2}}+2(r-1)^{\frac{3}{2}}-2}) \in V_1\) and \((v(0) = 0, v(2) = \frac{2r[(r-1)^{\frac{3}{2}}+1]}{3r+3r(r-1)^{\frac{3}{2}}+2(r-1)^{\frac{3}{2}}-2}) \in V_0\). The resulting aggregate effort levels generated by the two prize structure provide two lower bounds for the maximal effort inducible in \(V_1\). More calculations and details are included in appendix.

We next pin down the expected aggregate effort induced by a prize profile in \(V_2\) and \(V_3\) respectively in the following. For any given prize structure, computing aggregate effort requires solving the players’ choices of effort on each stage game for every possible path. Since the uniqueness of the equilibrium described by Lemma 1(ii) is not established for one-shot two-player Tullock contest with asymmetric valuations when \(r\) is higher than a cutoff, players’ equilibrium efforts on second stage contest are calculated using the only known equilibrium for a prize structure in \(V_2 \cup V_3\).
Lemma 7 \( \forall r \in (\bar{r}, 2] \), a prize profile in \( V_2 \) induces the aggregate effort

\[
TE_2 = \frac{r}{2}(1 - 2v(0)) + (1 - \frac{1}{r})(\frac{1}{r - 1})^{\frac{1}{2}}(2 - r)[\frac{r}{2} - (\frac{1}{2} + \frac{r}{4})v(0) + (\frac{1}{2} - \frac{3r}{4})v(2)].
\]

**Proof.** See appendix. ■

In the next two propositions, we show that we can ignore the prize profiles in \( V_2 \) and \( V_3 \) when searching for the optimal prize structure.

**Proposition 1** \( \forall r \in (\bar{r}, 2], \) we have \( \sup_{V_2} TE_2 = TE_2^* := \frac{r}{2} + \frac{(r-1)(2-r)}{[(r-1)^{\frac{1}{2}}(1+(\frac{1}{2} + \frac{3}{4})\frac{2}{r-1})]^2} \). However, there exists no prize structure in \( V_2 \) that can induce \( \sup TE_2 \).

**Proof.** To maximize effort \( TE_2 \) in (5) subject to prize profiles are in \( V_2 \), we first note that \( TE_2 \) decreases in \( v(2) \) and \( v(2) > \frac{r[1+(r-1)^{\frac{1}{2}}]}{[(r-1)^{\frac{1}{2}}(1+(\frac{1}{2} + \frac{3}{4})\frac{2}{r-1})]^2} \) holds within \( V_2 \). By calculation, the proposition follows. More details are in appendix. ■

Similar to Lemma 7 and Proposition 1, we provide the total effort induced by prize profiles in \( V_3 \), and establish an upper bound which is not attainable in \( V_3 \).

**Proposition 2** \( \forall r \in (\bar{r}, 2], \) a prize profile in \( V_3 \) induces the aggregate effort

\[
TE_3 = \frac{r}{2}(2v(2) - 1) + (1 - \frac{1}{r})(\frac{1}{r - 1})^{\frac{1}{2}}(2 + r)[\frac{r}{2} - (\frac{1}{2} - \frac{r}{4})v(0) + (\frac{1}{2} + \frac{3}{4})v(2)].
\]

We have \( \sup_{V_3} TE_3 = TE_3^* := \frac{(\frac{2}{3} + \frac{r}{2})(r-1)^{\frac{1}{2}} + \frac{5}{2}r^2 + \frac{5}{4} - 2}{[1 + \frac{5}{2}r + (\frac{3}{4} - 1)(r-1)^{\frac{1}{2}}]} \). Moreover, there exists no prize structure in \( V_3 \) that can induce \( \sup TE_3 \).

**Proof.** Analogously, we calculate the aggregate effort \( TE_3 \) and maximize \( TE_3 \) in (6) subject to prize profiles are in \( V_3 \). More details are in appendix. ■

**Proposition 3** \( \forall r \in (\bar{r}, 2], \) the optimal prize profile cannot be in \( V \backslash V_1 \) and it must be in \( V_1 \) if it exists.

**Proof.** According to Lemma 6, Propositions 1 and 2, we have \( \max_{V_1} TE_1 \geq \max \{ \sup_{V_2} TE_2, \sup_{V_3} TE_3 \} \). Together with Lemma 5, we have this proposition. ■

The next Theorem establishes the existence of optimal prize profile, which is explicitly characterized. Note that the discussion following Property 2 reveals that \( D(\frac{4}{3} - \frac{5}{2}, r) \) has a unique root \( r \) on \( [0, 2) \).

**Theorem 2** \( \forall r \in (\bar{r}, 2], \) the optimal prize profile exists and is in \( V_1 \). Let \( r^* \in (\bar{r}, 2) \) such that \( D(v = (r - 1)^{\frac{1}{2}}, r)|_{r=r^*} = 0 \). The effort-maximizing prize allocation \( (v(0), v(1), v(2), v(3)) \) is

(i) \( (0, 1 - v^*_r(2), v^*_r(2), 1) \), where \( v^*_r(2) \in [\frac{2}{3}, 1] \) is determined by \( D(\eta(v(0) = 0, v^*_r(2), r), r) = 0 \) if \( r \in (\bar{r}, r^*) \);

(ii) \( (0, 1 - v^*_r(2), v^*_r(2), 1) \), where \( v^*_r(2) = 1 - \frac{\frac{5}{2} + \frac{5}{4}r + (\frac{1}{2} + \frac{5}{4})r - 1}{\frac{1}{2} + \frac{5}{4}r + (\frac{1}{2} + \frac{5}{4})(r-1)^{\frac{1}{2}}} \in [\frac{2}{3}, 1] \) if \( r \in (r^*, 2] \).
Proof. From Proposition 3, it is sufficient to search the optimal prize structure within $\mathcal{V}_1$. By Lemma 4 and the proof of Theorem 1, we obtain that $v^*(0) = 0$ and $\frac{d}{d\eta}TE_1(v(0), v(2))|_{v(0) = 0} = D(\eta, r)$, where $D(\eta, r)$ is increasing in $\eta$ for each $r \in [\overline{r}, 2]$.

In $\mathcal{V}_1$, $r \leq 1 + \left(\frac{w_B}{w_A}\right)^2$ and $w_A \geq w_B$ leads to $-\frac{1}{2} + \frac{2}{3} + \left(\frac{1}{2} + \frac{1}{6}\right)(r-1)\frac{1}{r} \leq v(1)$ and $v(1) \leq \frac{1}{3}$ respectively, as a result, $v(2) \in \left[\frac{2}{3}, 1 - \frac{1}{2} + \frac{2}{3} + \left(\frac{1}{2} + \frac{1}{6}\right)(r-1)\frac{1}{r}\right]$. Note that $r \geq \overline{r}$ implies that $\eta := \frac{w_B}{w_A} \in \left[(r-1)^\frac{1}{r}, 1\right] \subset \left[\frac{2}{3}, 1\right]$.

Direct calculation yields that $\eta(v(0) = 0, v(2), r)$ is decreasing in $v(2)$ and thus $D(\eta(v(0) = 0, v(2), r), r)$ is decreasing in $v(2)$.

By the similar arguments in Theorem 1, $D(\eta, r)|_{\eta=(r-1)^\frac{1}{r}} > 0$ for each $r \in (r^*, 2]$, which implies an increase in $v(2)$ will raise the aggregate effort, so that $v^*(2) = 1 - \frac{1}{2} + \frac{2}{3} + \left(\frac{1}{2} + \frac{1}{6}\right)(r-1)\frac{1}{r}$ as this is the upper bound implied by $\mathcal{V}_1$.

Analogously, we have $D(\eta, r)|_{\eta=(r-1)^\frac{1}{r}} < 0$ for $r \in (\overline{r}, r^*)$, where $r^*$ satisfies $D((r-1)^\frac{1}{r}, r)|_{r=r^*} = 0$. Since $D(\eta(v(0), v(2), r), r)|_{v(0) = 0} > 0$ when $v(2) < v^*_r(2)$ and $D(\eta(v(0), v(2), r), r)|_{v(0) = 0} < 0$ when $v(2) > v^*_r(2)$, the optimal $v^*_r(2)$ is determined by $D(\eta(v(0), v(2), r)|_{(v(0), v(2))=(0, v^*_r(2))} = 0$ for each $r \in (\overline{r}, r^*)$.

The details are included in the appendix. $\blacksquare$

Theorem 2 can also be rewritten by using alternative setup as follow. In a sequential three battle contest with two players, expected aggregate effort is maximized only when $(v, V)$ equals (i) $(1 - v^*_r(2), 3v^*_r(2) - 2)$ if $r \in (\overline{r}, r^*)$, where $v^*_r(2) \in (\frac{2}{3}, 1)$ can be determined from $D(\eta(v(0), v(2), r), r)|_{v(0) = 0} = 0$ and thus $V = 3v^*_r(2) - 2 > 0$; (ii) $(1 - v^*_r(2), 3v^*_r(2) - 2)$ if $r \in (r^*, 2]$, where $v^*_r(2) = 1 - \frac{1}{2} + \frac{2}{3} + \left(\frac{1}{2} + \frac{1}{6}\right)(r-1)\frac{1}{r} \in \left(\frac{2}{3}, 1\right)$ and thus $V = 3v^*_r(2) - 2 \geq 0$.

Combining Theorem 1 and 2, the optimal prize allocation should satisfy $V > 0$ using alternative setup whenever $r \in [0, 2)$, which means the optimal prize structure should pay extra bonus to the player who has better aggregate performance in this case. Recall the equivalence between the alternative setup and the underlying setup, we can adopt the alternative setup $(V, v)$ with $V = v(2) - v(1) - (v(1) - v(0)) = 3v(2) - 2; v = v(1)$ to express the prize allocation $\{v(0); v(1); v(2); v(3)\}$ where $v^*(0) = 0$ and $v^*(3) = 1$. Consequently, $v(2)$ is a key to understand how the optimal prize structure depends on the players’ aggregate performance as contest technology changes in the underlying setup.

From now on, we focus on the prize $v(2)$. Before we proceed to access the effect of $v(2)$ on the aggregate effort, we first think about how the aggregate effort reacts with a change in $v(2)$. Consider an increase in $v(2)$, it provides better incentive to the winner of the first battle, but discourages the loser of the first battle more as the budget is fixed. Consequently, the contest becomes more imbalance in the sense that the early winner is more likely to win the second battle. However, both players turn to be willing to exert larger efforts in the first battle for this advantage. Note that the magnitudes of the effects we mention above all depend on $r$.

Therefore, we now evaluate the effect of $v(2)$ on the aggregate effort for any $r \in (0, 2]$ to make this
trade-off clearer. Recall the proof of Theorem 1 and 2, we have \( \frac{d}{dv(2)} TE_1(v(0), v(2))|_{v(0)=0} \) = \( D(\eta, r) \), where

\[
\eta \equiv w_B \equiv \frac{w_B}{w_A} = \left( \frac{1}{2} + \frac{r}{4} \right)(1 - v(2)) + \left( \frac{1}{2} - \frac{r}{4} \right)(2v(2) - 1)
\]

\( \frac{1}{2} - \frac{r}{4} \) can be regarded as a function of \( v(2) \) and \( r \).

We first note that \( D(\eta, r) \) is a decreasing function of \( v(2) \) for each positive \( r \). This is because \( D(\eta, r) \) is increasing in \( \eta \) by Property 3 and \( \eta \) is decreasing function of \( v(2) \) for each \( r \).

We then consider two lower bounds of \( \eta \). One lower bound implied its definition is \( \frac{1}{2} - \frac{r}{4} \), which can be achieved by a feasible prize structure when \( r \in (0, \pi) \). Within \( V_1 \), the other one is \( (r - 1)^\frac{1}{2} \), which can be achieved when \( r \in (\pi, 2) \). We thus define \( \eta_{lower} := \max\{\frac{2 - r}{2 + r}, (r - 1)^\frac{1}{2}\} \).

When \( D(\eta, r)|_{\eta=\eta_{lower}} > 0 \), we have \( \frac{d}{dv(2)} TE_1(v(0), v(2))|_{v(0)=0} > 0 \) for all feasible prize structures, that is, an increasing in \( v(2) \) can raise the aggregate effort so that \( v(2) \) should be as high as possible at the optimum. In other words, the optimal \( v(2) \) is achieved at the boundary.

While \( D(\eta, r)|_{\eta=\eta_{lower}} < 0 \), we have \( \frac{d}{dv(2)} TE_1(v(0), v(2))|_{v(0)=0} < 0 \) for the highest feasible \( v(2) \), so that reducing \( v(2) \) can increase the aggregate effort until \( D(\eta(v(0), v(2)), r) = 0 \). In this case, we have interior optimum.

In sum, \( D(\eta, r) \) summarizes the effect of a change in \( v(2) \) on the aggregate effort.

In the following Figure 1, we plot \( D(\eta, r)|_{\eta=\eta_{lower}} = D(\eta, r)|_{\eta=\max\{\frac{2 - r}{2 + r}, (r - 1)^\frac{1}{2}\}} \) when \( r \in (0, 2) \).

![Figure 1: D(\eta, r)|_{\eta=\eta_{lower}}](image)

In the graphs, \( r^* \approx 1.09 \) is the horizontal ordinate of the intersection of the black line and the blue line; \( r^* \approx 1.31 \) is the horizontal ordinate of the intersection of the red line and the black line. If \( r \)}
is between $r$ and $r^*$, the optimal $v_r^*(2)$ is in the interior and determined from $D(\eta, r)$; otherwise, the optimal $v_r^*(2)$ is at the boundary and should equal the highest feasible value of $v(2)$.

Figure 2 plots the optimal prize $v_r^*(1)$ as a function of $r$, that is,

$$v_r^*(1) = \begin{cases} 
0 & \text{when } r \in (0, r] \\
1 - v_r^*(2), \text{ where } v_r^*(2) \text{ is determined from } D(\eta(v(0), v(2)), r) |_{v(0)=0} = 0 & \text{when } r \in (r, r^*) \\
-\frac{1}{2} + \frac{r}{4} + (\frac{1}{2} + \frac{r}{4})(r-1)^2 & \frac{1}{2} + \frac{3r}{4} + (\frac{1}{2} + \frac{3r}{4})(r-1)^2 & \text{when } r \in [r^*, 2]
\end{cases}$$

Note that by Figure 2, $v_r^*(1)$ strictly increases with $r \in [r, 2]$ and is bounded above by $\frac{1}{3}$.

![Figure 2: the optimal prize $v_r^*(1)$](image)

The part of curve in red stands for the optimal prize $v_r^*(1)$ which is an interior optimum when $r \in (r, r^*)$. The rest of optimal prize $v_r^*(1)$ should be as lower as possible since $v(1) = 1 - v(2)$. The black line means that $v_r^*(1) = 0$ when $r \in (0, r]$. While the part of curve in blue stand for $-\frac{1}{2} + \frac{r}{4} + (\frac{1}{2} + \frac{r}{4})(r-1)^2$ which is a lower bound imposed by $\mathcal{V}_1$.

### 3.3 Case 3: $r > 2$

We now consider the remaining case of $r > 2$, in which all prize profiles are covered by sets $\mathcal{V}_4$ and $\mathcal{V}_5$. In Lemmas 8 and 9, we provide the expected aggregate effort under the prize profiles in $\mathcal{V}_4$ and $\mathcal{V}_5$ respectively. In Propositions 4 and 5, we characterize the optimal prizes in $\mathcal{V}_4$ and $\mathcal{V}_5$ respectively. Combining both propositions fully pins down the optimal prize structures when $r > 2$.

For a given prize structure, computing players’ aggregate effort requires solving the players’ choices of effort on each stage game for every possible path. Since Hillman and Riley (1989) and Baye, Kovenock,
and de Vries (1996) verify the existence of a unique mixed equilibrium described by Lemma 1(iii) in one-shot all-pay contest, the corresponding equilibrium is unique on each stage game, the usual backwards induction argument shows that the subgame perfect equilibrium of the whole contest is unique. While players’ efforts on each game are calculated using the only know equilibrium though uniqueness is not established for Tullock contest with $r > 2$.

**Lemma 8** When $r > 2$, the expected aggregate effort $TE_4 = 1 - 2v(0)$ for all prize profiles in $\mathcal{V}_4 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \geq v(1) - v(0)\}$.

**Proof.** Note that when $r > 2$, we have Lemma 1(iii) applies. As usual, we solve the game by backward induction.

First, consider battle 3. at history $(n_A, n_B) = (2, 0)$: For history $(2, 0)$, the common effective prize spread is:

$$v_A(2, 0) = v_B(2, 0) = v(1) - v(0) \geq 0.$$ 

Thus effort supply (in the mixed-strategy equilibrium) is given by

$$\tilde{x}_A(2, 0) \sim G^A_{(2,0)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)],$$

$$\tilde{x}_B(2, 0) \sim G^B_{(2,0)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)],$$

where $G^i_{(n_A, n_B)}(\cdot), i = A, B$ denotes the cumulative distribution function of player $i$.

The winning probabilities are

$$p_A(2, 0) = p_B(2, 0) = \frac{1}{2}.$$ 

History $(0, 2)$ is symmetric. The effective prize spread is

$$v_A(0, 2) = v_B(0, 2) = v(1) - v(0) \geq 0.$$ 

The effort supply is

$$G^A_{(0,2)}(x) = G^B_{(0,2)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)].$$

The winning probabilities are

$$p_A(0, 2) = p_B(0, 2) = \frac{1}{2}.$$ 

When $(n_A, n_B) = (1, 1)$, the common effective prize spread is

$$v_A(1, 1) = v_B(1, 1) = v(2) - v(1) \geq 0.$$ 

Thus effort supply is

$$G^A_{(1,1)}(x) = G^B_{(1,1)}(x) = \frac{x}{v(2) - v(1)} \text{ in } [0, v(2) - v(1)].$$
The probabilities are

\[ p_A(1, 1) = p_B(1, 1) = \frac{1}{2}. \]

We now consider the second battle. When \((n_A, n_B) = (1, 0)\), the effective prize spreads are as follows. For player 1,

\[
\tilde{v}_A(1, 0) = [p_A(2, 0)v(3) + p_B(2, 0)v(2) - E\tilde{x}_A(2, 0)] - [p_A(1, 1)v(2) + p_B(1, 1)v(1) - E\tilde{x}_A(1, 1)]
\]

\[ = \frac{1}{2}(v(3) - v(1)) + E\tilde{x}_A(1, 1) - E\tilde{x}_A(2, 0) \]

\[ = v(2) - v(1). \]

Similarly,

\[ \tilde{v}_B(1, 0) = v(1) - v(0). \]

We have \(\tilde{v}_A(1, 0) \geq \tilde{v}_B(1, 0)\) if and only if \(v(2) - v(1) \geq v(1) - v(0)\), which is the case considered in this lemma, i.e. prize profiles in \(V_4\).

Thus the effort supply is

\[ G_A^{(1, 0)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, (v(1) - v(0))], \]

\[ G_B^{(1, 0)}(x) = \frac{(v(2) - v(1)) - (v(1) - v(0)) + x}{(v(2) - v(1))} \text{ in } [0, v(1) - v(0)]. \]

The winning probabilities are

\[ p_A(1, 0) = 1 - \frac{1}{2}q, \quad p_B(1, 0) = \frac{q}{2}, \]

where \(q = \frac{v(1) - v(0)}{v(2) - v(1)}. \)

Players’ expected effort is

\[ E[\tilde{x}_A(1, 0)] = \frac{1}{2}(v(1) - v(0)), \quad E[\tilde{x}_B(1, 0)] = \frac{1}{2} \frac{(v(1) - v(0))^2}{(v(2) - v(1))}. \]

History \((0, 1)\) is symmetric. We have effort supply

\[ G_A^{(0, 1)}(x) = G_B^{(1, 0)}(x), \quad G_B^{(0, 1)}(x) = G_A^{(1, 0)}(x). \]

The winning probabilities are

\[ p_A(0, 1) = \frac{1}{2}q, \quad p_B(0, 1) = 1 - \frac{1}{2}q. \]
Now we come to the first battle. The common effective prize spread is

\[ v_A(0, 0) = v_B(0, 0) = \{ p_A(1, 0)[p_A(2, 0)v(3) + p_B(2, 0)v(2) - E[\tilde{x}_A(2, 0)] + p_B(1, 0)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - E[\tilde{x}_A(1, 1)] - E[\tilde{x}_A(1, 0)] - \{ p_A(0, 1)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - E[\tilde{x}_A(1, 1)] + p_B(0, 1)[p_A(0, 2)v(1) + p_B(0, 2)v(0) - E[\tilde{x}_A(0, 2)] - x_A(0, 1)\}

= (1 - \frac{1}{2}q)(v(2) - v(0)) + (-\frac{1}{2} + \frac{q}{2})(v(1) - v(0)). \]

Thus the effort supply is

\[ G_A^{(0, 0)} = G_B^{(0, 0)} = \frac{x}{v_A(0, 0)} \text{ in } [0, v_A(0, 0)], \]

and the winning probabilities are

\[ p_A(0, 0) = p_B(0, 0) = \frac{1}{2}. \]

Total effort thus is as follow:

\[ TE_4 = 2E[\tilde{x}_A(0, 0)] + E[\tilde{x}_A(1, 0)] + E[\tilde{x}_B(1, 0)] + p_A(1, 0)(E[\tilde{x}_A(2, 0)] + E[\tilde{x}_A(2, 0)]) + p_B(1, 0)(E[\tilde{x}_A(1, 1)] + E[\tilde{x}_A(1, 1)])
\]

\[ = v_A(0, 0) + \frac{1}{2}(1 + q)(v(1) - v(0)) + (1 - \frac{q}{2})(v(1) - v(0)) + \frac{q}{2}(v(2) - v(1))
\]

\[ = 2(v(1) - v(0)) + (v(2) - v(1))
\]

\[ = v(2) + v(1) - 2v(0)
\]

\[ = 1 - 2v(0). \]

\[ \text{Proposition 4} \quad \text{When } r > 2, \text{ maximum aggregate effort in } \mathcal{V}_4 \text{ equals } 1 \text{ and is supported by any prize profile } (0, 1 - v(2), v(2), 1) \text{ for which } v(2) \in [\frac{2}{3}, 1]. \]

\[ \text{Proof.} \quad \text{By Lemma 8, maximizing the total effort } TE_4 \text{ among prize structures in } \mathcal{V}_4 \text{ yields the optimal allocations } v(0) = 0, v(2) \in [\frac{2}{3}, 1], v(1) = 1 - v(2), \text{ and } v(3) = 1. \text{ As a result, } TE_4^* = 1, \text{ i.e. the rent is fully dissipated.} \]

Following similar procedure, we obtain the effort supply as in the following lemma when prize profiles are in \( \mathcal{V}_5 \). To save space, the proof is relegated to the appendix.

\[ \text{Lemma 9} \quad \text{When } r > 2, \text{ the expected aggregate effort is } TE_5 = 3(v(2) - v(1)) \text{ for all prize structures in } \mathcal{V}_5 = \mathcal{V} \cap \{(v(0), v(2)) : v(2) - v(1) \leq v(1) - v(0)\}. \]
Proof. See appendix. ■

Proposition 5  When \( r > 2 \), maximum aggregate effort equals 1 in \( V_5 \) is supported by the prize structure \((0, \frac{1}{2}, \frac{2}{3}, 1)\).

Proof. \( TE_5 = 3(v(2) - v(1)) \) by Lemma 9. Therefore the designer’s problem is

\[
\begin{align*}
\max & \quad TE_5 \\
\text{s.t.} & \quad (v(1) - v(0)) \geq (v(2) - v(1)), \\
& \quad 0 \leq v(0) \leq \frac{1}{2} \leq v(2) \leq 1, \\
& \quad v(2) + v(0) \leq 1,
\end{align*}
\]

which yields \( v(0) = 0, v(1) = \frac{1}{3}, v(2) = \frac{2}{3} \) and \( v(3) = 1 \). For this prize profile, we have \( TE_5^* = 1 \). ■

Note that \( V = V_4 \cup V_5 \) when \( r > 2 \). Combining Propositions 4 and 5 immediately gives the following theorem, which fully characterizes the optimal prize profile when \( r > 2 \).

Theorem 3  When \( r > 2 \), maximum aggregate effort equals 1 is supported by any prize structure \((0, 1 - v(2), v(2), 1)\) for which \( v(2) \) can be any value between \([\frac{2}{3}, 1]\).

3.4 Intuitions

We next illustrate the economics and intuitions behind these characterizations in Theorems 1 to 3. It is natural that \( v(0) = 0 \) (and thus \( v(3) = 1 \)) is necessary to elicit maximal effort from players as rewarding a player without a single win definitely dampens the players’ incentive. The more interesting and intricate trade-off lies in the balance between \( v(1) \) and \( v(2) \), the prizes for a single win and two wins. In other words, should a positive prize be granted to a player with a single win? If yes, how much prize should be awarded to a single win at the optimum?

An \( \varepsilon(>0) \) increase in \( v(1) \) generates different marginal impacts on the incentive of players through changes in effective prize spreads in each battle upon different contingencies of early battle outcomes. Moreover, this change in \( v(1) \) affects the probability of each contingency is reached in the equilibrium path.

3.4.1 Battle 3

When one player wins both of the first two battles, a change in \( v(1) \) would increase the effective prize spread by \( \varepsilon \) in the third battle, which leads to higher effort supply. When each of the two players wins one battle of the first two, such a change in \( v(1) \) reduces the effective prize spread by \( 2\varepsilon \) in the third battle, which leads to lower effort supply. Both effects increase with the discriminatory power \( r \) since the third battle always has a symmetric prize spread though it depends on the outcomes of the first two battles. Therefore, fix the winning probabilities in the second battle, the positive effect would dominate the negative effect if and only if the probability of the event that one player wins the first two battles
is twice higher than the probability that each player wins one of the first two battles. In other words, the increase in $v(1)$ would generate a positive impact if and only if the prevailing momentum effect in the second battle is sufficiently strong.

Moreover, the marginal impact of change in $v(1)$ on the second battle winning probabilities, i.e. the momentum effect, rather plays an important role when determining its impact on the third battle effort supply as different levels of effort are induced in the subsequent subgames. For $v(1) < \frac{1}{3}$, the subgame starting with a history that each player wins one early battle induces higher effort than the subgame starting with a history that one player wins both earlier battles, since the former subgame has a higher prize spread. It follows that the mitigation of the momentum effect due to a higher $v(1)$ would definitely generate a positive effect on the effort supply in the third battle. This impact will be further discussed in Section 3.4.2 when we talk about battle 2.

A higher discriminatory power $r$ is associated with a stronger prevailing momentum effect in the second battle. At the same time, with a higher $r$, the momentum mitigation effect of an increase in $v(1)$ is more significant. We thus expect that a higher $v(1)$ would generate higher effort supply from the third battle if and only if the discriminatory power $r$ is high enough.

### 3.4.2 Battle 2

How does the above $\varepsilon$ increase in $v(1)$ affect the prize spreads for both players in the second battle and their effort supply in this battle? Moreover, how is the momentum effect in the second battle affected by the change in $v(1)$?

We first consider the winner of the first battle. The change in his effective prize spread is determined by the changes in his payoffs when he wins both the first two battles and when he wins only one of the first two battles. On one hand, the expected payoff of a player conditional on his winning the first two battles must drop since the expected prize decreases but effort cost increases due to increased prize spread in the third battle.\(^\text{13}\) On the other hand, the expected payoff of a player conditional on his winning only one of first two battles must increase since the expected prize does not change but effort cost decreases due to decreased prize spread in the third battle.\(^\text{14}\) Therefore, an $\varepsilon$ increase in $v(1)$ reduces the prize spread in the second battle for the winner of the first battle.

We now turn to the loser of the first battle. On one hand, we have explained above that the expected payoff of a player conditional on his winning only one of first two battles must increase. On the other hand, the expected payoff of a player conditional on his losing the first two battles must also weakly increase since the expected prize increases though effort cost also increases due to increased prize spread in the third battle.\(^\text{15}\) Therefore, the impact of an $\varepsilon$ increase in $v(1)$ on the second-battle prize spread of the loser of the first battle can be ambiguous. One can verify that for $r \geq 2$, we have a prize spread

\(^{13}\) At the unique symmetric equilibrium, the concerned player wins $v(3)$ and $v(2)$ with same probability of $\frac{1}{2}$.

\(^{14}\) At the unique symmetric equilibrium, each player wins $v(1)$ and $v(2)$ with same probability of $\frac{1}{2}$. Note $v(1) + v(2) = 1$.

\(^{15}\) At the unique symmetric equilibrium, the concerned player wins $v(0) = 0$ and $v(1)$ with same probability of $\frac{1}{2}$. When $r < 2$, the rent $v(1)$ is not fully dissipated, and concerned player’s payoff strictly increases with $v(1)$. When $r \geq 2$, the rent is fully dissipated, concerned player’s payoff is fixed at zero.
$v(1)$, which increases with $v(1)$; For $r = 0$, we have a prize spread of $v_r(2)$, which decreases with $v(1)$. Due to continuity of the prize spread as a function of $r$, we reasonably expect that the second battle prize spread of a loser of the first battle decreases with $v(1)$ when $r$ is low, but increases with $v(1)$ when $r$ is high.

The above illustrated changes in players’ second battle prize spread mean that a high $r$ shrinks the gap between players’ second battle prize spreads, which further reduces the momentum effect. This effect enlarges the positive impact of $v(1)$ on expected effort supply in the third battle. When $r$ is small, both players’ second battle prize spreads drop with $v(1)$, the change in their difference is less clear. It is possible that the momentum effect can even be enhanced by an increase in $v(1)$, which contributes to its negative impact on expected effort supply in the third battle.

With a high $r$, the reduced gap between players’ prize spreads means a more even battle field in battle 2, which tends to induce more effort supply provided that the total prize spreads does not vary much. With a low $r$, both players’ second-battle prize spreads decrease, which tends to reduce the effort supply in the second battle.

3.4.3 Battle 1

We now turn to the first battle. How does the above $\varepsilon$ increase in $v(1)$ affect the common prize spread in the first battle? We first look at how the expected payoffs of the winner and the loser of the first battle are affected. When $r$ is high, due to the reduced momentum effect, the winner of the first battle would win less expected prize; and the loser of the first battle would win more expected prize. On the other hand, both of their second stage and third stage expected effort supply tend to be higher. Therefore, the expected payoff of the winner of the first battle is lower. While the change in the expected payoff of the loser of the first battle cannot be immediately pinned down. It is reasonable to expect that an increase in $v(1)$ likely eventually benefits him. Therefore, with high $r$, the first battle prize spread should be lower, which renders lower first battle effort supply.

When $r$ is low, whether the momentum effect is reduced or enhanced is not clear, which makes it hard to pin down the signs of changes in the expected prizes and expected effort supply (both second stage and third stage) for both the winner and loser of the first battle. Nevertheless, intuitively an increase in $v(1)$ would most likely hurt more the winner and benefit more the loser of the first battle. Therefore, one can expect a lower first battle prize spread results, which means lower first battle effort supply.

Based on the above discussions, the first battle most likely generates less total effort with an $\varepsilon$ increase in $v(1)$ regardless of the magnitude of $r$. However, a high $r$ would tend to generate higher effort from the second and third battle while the impact of a lower $r$ on the effort generated from the second and third battle can be ambiguous. These observations rationalize how the optimal allocation should reply on $r$. When $r$ is small (i.e. when $r \leq r_0$), the ambiguous effect of an $\varepsilon$ increase in $v(1)$ on the second and third battle is dominated by its negative effect on the first battle effort supply, which leads to the optimality of a zero $v(1)$. When $r$ is big (i.e. when $r > r_0$), the positive effect of an $\varepsilon$ increase in $v(1)$
on the second and third battle at least offsets its negative effect on the first battle effort supply, which leads to the optimality of a positive \( v(1) \). When \( r \in (r, 2) \), the direct positive impact on the second and third battle is more likely to dominate the more indirect negative impact on the first battle effort supply, we thus have \( v(1) \) increases with \( r \) in this range. When \( r \geq 2 \), we have that each battle reaches its maximal efficiency in eliciting effort supply. In particular, the rents are fully dissipated in the first battle in which the two players' prize spreads are symmetric. Higher \( v(1) \) increases effort supply in the second and third battle, however, the change in the effort supply is fully offset by the effort decrease in the first battle due to the full surplus extraction in the first stage, as long as the strategic momentum for the winner of the first battle still persists.

These intuitions can be further illustrated after we reinterpret the prize structure in the next subsection using the language of grand contest prize and battle prizes.

### 3.5 Alternative interpretation: battle prize and grand contest prize

The above identified optimal prize structures can be conveniently interpreted using battle prizes and grand contest prize. A player gets a battle prize \( v_B \) whenever he wins an individual battle and a grand winner who wins at least two battles gets a grand contest prize or a punishment \( v_G \). Under this alternative prize structure, a player receives a prize \( v(0) \) if he wins no battle; he receives a prize \( v(0) + v_B \) if he wins a single battle; he receives a prize \( v(0) + 2v_B + v_G \) if he wins two battles; and he receives a prize \( v(0) + 3v_B + v_G \) if he wins all three battles.

For any eligible prize structure \( \{v(0); v(1); v(2); v(3)\} \) that satisfies the three restrictions (1), (2) and (3) in the original setup, we next identify a unique prize structure \( (v(0), v_B, v_G) \) satisfying the budget constraint \( 3v_B + v_G = 1 - 2v(0) \) such that this alternative prize structure generates the identical effective rewards contingent on number of wins. For this purpose, we let \( v_B := v(1) - v(0) \) and \( v_G := v(2) - v(1) - (v(1) - v(0)) \).

Note that the corresponding budget constraint is satisfied automatically:

\[
3v_B + v_G = 2v(0) + 3(v(1) - v(0)) + v(2) - v(1) - (v(1) - v(0)) = v(2) + v(1) - 2v(0) = 1 - 2v(0).
\]

We next show that each player faces same prize spread across the two prize structures at every state of the contest. As a result, each player exerts same effort across the two setups at each contingency along the path of the contest.

We first look at battle 3. If a player secures zero win, his prize spread is \( v(1) - v(0) \) in the original setup, while his prize spread is \( v_B \) in the alternative setup, and the two coincide; if a player secures one win, his prize spread is \( v(2) - v(1) \) in the original setup, while his prize spread is \( v_B + v_G \) in the alternative setup, and the two coincide; if a player secures two wins, his prize spread is \( v(3) - v(2) \) in the original setup, while his prize spread is \( v_B \) in the alternative setup, and the two coincide. The equivalence in the prize spreads across the two prize structures means that the effort supply and winning chances are identical on every contingency in battle 3.

The above discussion further means that the prize spreads in battle 2 must be same across the two

---

\[16\] Note that when \( v(0) = 0 \), we have \( v_G < 0 \) if and only if \( v(1) > \frac{1}{3} \), but it never occurs for the optimal prize allocation where \( v(1) \leq \frac{1}{3} \).
prize structures. Similar argument applies to battle 1. More precisely, in battle 2, for a player who wins
the first battle, his prize spread in the original setup is
\[ \begin{align*}
v_A(1, 0) &= v_B(0, 1) \\
&= \left[ \frac{1}{2} v(3) + \frac{1}{2} v(2) - E_x A(2, 0) \right] - \left[ \frac{1}{2} v(2) + \frac{1}{2} v(1) - E_x A(1, 1) \right] \\
&= \frac{1}{2} v(3) - \frac{1}{2} v(1) - E_x A(2, 0) + E_x A(1, 1).
\end{align*} \]

Since effort supply \( E_x A(2, 0) \) and \( E_x A(1, 1) \) are same across the two prize structures in battle 3, his
prize spread in the alternative setup is
\[ \begin{align*}
\tilde{v}_A(1, 0) &= \tilde{v}_B(0, 1) \\
&= \frac{1}{2} (v(0) + 3v_B + v_G) + \frac{1}{2} (v(0) + 2v_B + v_G) - E_x A(2, 0) \\
&\quad - \left[ \frac{1}{2} (v(0) + 2v_B + v_G) + \frac{1}{2} (v(0) + v_B) - E_x A(1, 1) \right] \\
&= \frac{1}{2} (2v_B + v_G) - E_x A(2, 0) + E_x A(1, 1) = \frac{1}{2} v(3) - \frac{1}{2} v(1) - E_x A(2, 0) + E_x A(1, 1).
\end{align*} \]

Analogously, the prize spread of the player who loses the first battle remain the same across the two
prize structures. Consequently, the players’ winning probabilities and effort supply are same in battle
2 under the two setups.

We now turn to battle 1. Note in battles 2 and 3, the equilibrium efforts are the same across the
two prize structures on every contingency along the path. The two players in the original setup have a
common prize spread
\[ \begin{align*}
v_A(0, 0) &= v_B(0, 0) \\
&= \{ p_A(1, 0)[p_A(2, 0)v(3) + p_B(2, 0)v(2) - E_x A(2, 0)] \\
&\quad + p_B(1, 0)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - E_x A(1, 1)] - E_x A(1, 0) \} \\
&\quad - \{ p_A(0, 1)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - E_x A(1, 1)] \\
&\quad + p_B(0, 1)[p_A(0, 2)v(1) + p_B(0, 2)v(0) - E_x A(0, 2)] - E_x A(0, 1) \} \\
&= p_A(1, 0)[p_A(2, 0)v(3) + p_B(2, 0)v(2) - E_x A(2, 0)] - E_x A(1, 0) \\
&\quad - p_B(0, 1)[p_A(0, 2)v(1) + p_B(0, 2)v(0) - E_x A(0, 2)] + E_x A(0, 1) \\
&= p_A(1, 0)\frac{1}{2} v(3) + \frac{1}{2} v(2) - \frac{1}{2} v(1) - \frac{1}{2} v(0) - E_x A(2, 0) + E_x A(0, 2)] - E_x A(1, 0) + E_x A(0, 1) \\
&= p_A(1, 0)[1 - v(1) - v(0) - E_x A(2, 0) + E_x A(0, 2)] - E_x A(1, 0) + E_x A(0, 1). 
\end{align*} \]
In the alternative setup, the same procedure renders their common prize spread

\[ \bar{v}_A(0, 0) = \bar{v}_B(0, 0) = p_A(1, 0)[1/2(3v_B + v_G) + 1/2(2v_B + v_G)] \]

\[ = p_A(1, 0)[1/2v_B - Ex_A(2, 0) + Ex_A(0, 2)] - Ex_A(1, 0) + Ex_A(0, 1) \]

by the constructions of \( v_B, v_G \). Therefore, the prize spreads are same across the two prize structures in battle 1.

We thus have the following implementation result for the optimal prize structure identified in Theorems 1 to 3.

**Theorem 4** \( \forall r > 0 \), the optimal design can be implemented by a sequential best-of-three contest with a battle prize \( v_B^* = v^*(1) \) for each battle and a grand contest prize \( v_G^* = 1 - 3v^*(1) \).

The economics and intuitions behind Theorem 4 can be illustrated as follows. It is natural that \( v^*(0) = 0 \) (and thus \( v^*(3) = 1 \)) is necessary to elicit maximal effort from players as rewarding a player without a single win definitely dampens the players’ incentive. According to the above prize structure equivalence result, any eligible prize structure \( \{v(n), n = 0, 1, 2, 3\} \) with \( v(0) = 0 \) and \( v(3) = 1 \) is then equivalent to a combination of a grand contest prize \( v_G \) to the grand winner who wins at least two battles and a uniform battle prize \( v_B \) to the winner of each battle. In particular, we have \( v_B = v(1) \) and \( v_G = 1 - 3v(1) \). A \( \Delta(> 0) \) increase in battle prize \( v_B \) thus must come together with a three-time drop in contest prize \( v_G \). These changes in the contest prize and battle prize have opposite impacts on the total effort induced. The decrease in contest prize would definitely lower the total effort supply. The increase in battle prizes would enhance effort supply through multiple channels. First, the increase in battle prize directly leads to higher effective prize spreads in component battles, which contribute to higher effort supply in each battle. Second, a higher battle prize mitigates the well established discouragement effect in the second battle, which increases the chance that the final winner is determined only after the third battle is fought. The mitigated discouragement effect thus tends to enhance the incentive provided by the contest prize. We would like to emphasize that this mitigation effect gets stronger for higher discriminatory power \( r \), i.e. when effort is more effective in determining the winner.

These discussions reveal that when discriminatory power \( r \) is low, the positive effect from a higher battle prize tends to be small. In this case, the negative effect of a lower contest prize tends to dominate, which leads to the optimality of zero battle prize. As the discriminatory power \( r \) moves into the intermediate range, the mitigation in the discouragement effect gets stronger, which renders the optimality of a positive battle prize that increases with \( r \). When \( r \) moves into a higher range, i.e. \( r \geq 2 \), the rents are fully dissipated in the first battle in which the two players’ prize spreads are symmetric. As a result, any eligible prize structure renders the same level of total effort supply.

Based on Theorems 1 to 4, we have the following observation regarding the optimal prize allocation.
rule, which says that proportional division rule is in general not optimal. In other words, in multi-battle contests, the optimal prize allocation rule should in principle give additional award to the grand winner of the whole contest.

**Corollary 1** \( \forall r > 0, \) we always have the optimal \( v^*_G \in [0, 1] \). Moreover, \( v^*_G > 0 \) unless \( r \geq 2 \).

## 4 Team contests with pairwise battles

In Sections 3, we have analyze a three-battle contest with two players. In this section, we analyze a three-battle contest with two teams with size of three. The team contests are sequentially played. In either case, each battle is fought between players from opposing teams and each player from a team plays only for one battle. Different from sequential contest with two players, in this team contest, a team player cares only about the final reward and cost in the current battle. And the team prize is a public good among team members.

This environment of teams contests with multiple pairwise battles has been first studied in Fu, Lu and Pan (2015) for equilibrium analysis purpose for a large class of contest technology of homogeneity-of-degree-zero while adopting the winner-take-all prize allocation rule. Häfner (2012) studies equilibrium analysis in a tug-of-war variant of this environment while assuming all-pay-auction technology and winner-take-all prize allocation rule.

We assume the same prize allocation framework as specified in Section 2: \( v(n), n \in \{0, 1, 2, 3\} \) denotes the prize allocated to the winning team. All three players in a team evaluate the prize at \( v(n) \), i.e. the prize is a public good within a team. Alternatively, the players within a same team equally split the prize their team wins. This alternative optimal prize sharing rule does not be affect the optimal prize allocation.

Based on the insight of Fu, Lu and Pan (2015), both involved players have a common prize spread in each battle, regardless of the previous outcome. However, depending on the magnitude of discriminatory power \( r \), the equilibrium bidding strategy takes two different forms. Below we show that pure equilibrium is described by Lemma 1(i) in the first case, and mixed equilibrium by Lemma 1(iii) in the second. The procedure for deriving the subgame perfect equilibrium is standard, and thus is relegated to the appendix.

Note that the uniqueness of the pure equilibrium described by Lemma 1(i) for Tullock contest with \( r \leq 2 \) and the mixed equilibrium described by Lemma 1(iii) for all pay contest is well established, the corresponding equilibrium is unique on each stage game, the usual backwards induction argument shows that the subgame perfect equilibrium of the whole contest is unique under each prize structure. While players’ equilibrium efforts on each stage game are calculated using the only know equilibrium described by Lemma 1(iii) for Tullock contest with \( r > 2 \) though uniqueness is not established.

### 4.1 Case 1: \( r \leq 2 \)

**Lemma 10** When \( r \leq 2 \), the expected total effort is \( TE^1_3 = \frac{2r}{4}(v(2) - v(0)) \) for the team contest.
Proof. See appendix.

Theorem 5 When \( r \leq 2 \), winner-take-all (i.e. \( (v(3), v(2), v(1), v(0)) = (1, 1, 0, 0) \)) uniquely maximizes aggregate effort.

Proof. Maximizing the total effort \( TE_S^1 = \frac{3r}{4}(v(2) - v(0)) \) given by Lemma 10 under required constraints for \( v(n) \) yields \( v(2) = v(3) = 1 \) and \( v(0) = v(1) = 0 \) in this case.

As for the uniqueness of the optimal prize structure, we proceed analogous to the proof of Theorem 1. Under each feasible prize structure, since the corresponding equilibrium is unique for each stage contest, the usual backwards induction argument shows that the subgame perfect equilibrium of the whole contest is unique. Consequently, the uniqueness of the optimal prize structure follows from the direct maximization above.

4.2 Case 2: \( r > 2 \)

Lemma 11 When \( r > 2 \), the expected total effort is \( TE_S^2 = \frac{3}{4}(v(2) - v(0)) \) for the team contest.

Proof. See appendix.

Theorem 6 When \( r > 2 \), winner-take-all (i.e. \( (v(3), v(2), v(1), v(0)) = (1, 1, 0, 0) \)) maximizes aggregate effort.

Proof. Maximizing the total effort \( TE_S^2 = \frac{3}{4}(v(2) - v(0)) \) given by Lemma 11 under required constraints for \( v(n) \) yields \( v(2) = v(3) = 1 \) and \( v(0) = v(1) = 0 \) in this case.

We thus see that optimal prize allocation rule can be different across contests played between the same two players and two teams of multiple players. When the discriminatory power \( r \) is low (i.e. \( r \leq \bar{r} \)), both two forms of contests require winner-take-all as the optimal prize allocation rule. When discriminatory power \( r \) is in the middle range (i.e. \( \bar{r} < r < 2 \)), team contest still requires a winner-take-all prize allocation for effort maximization while the contest between two same individual players requires awarding positive prizes to a player winning even a single battle. When discriminatory power \( r \) is in the high range (i.e. \( r \geq 2 \)), a winner-take-all prize allocation rule still uniquely maximizes the total effort supply in team contest while a wide span of prize allocation rules ranging from winner-take-all to proportional division rule is optimal in a contest between two same individual players.

The analysis on team contest further reinforces the insight that it is the mitigation of the discouragement effect that leads to the optimality of battles prizes in contests between two same individuals.

5 Concluding remarks

In this paper, we completely characterize the optimal contingent prize allocation rule in sequential-play three-battle contests between two same players, or two teams each with three players. The full spectrum of contest technologies in the Tolluck family are accommodated in our analysis. The optimal design can be implemented by a best-of-three contest with uniform battle prizes and a grand contest prize.
A winner-take-all best-of-three (a party wins all prize money if he wins more than two battles) induces the maximal total expected effort for contests between teams. When the battles are between two same individual players, the discriminatory power of the contest technology plays a crucial role in determining the optimal prize allocation rule. Specifically, when discriminatory power is in the low range, a winner-take-all best-of-three contest remains optimal. When the discriminatory power is in the intermediate range, the optimal battle prize strictly increases with the discriminatory power but never goes beyond one-third of the total prize. When the discriminatory power falls in the high range, a wide range of allocation rules in between winner-take-all and proportional division induces the maximal total expected effort.

The difference in the optimal prize structures across the two contest environments reflects the different dynamics in multi-battle contests between two individuals and two teams. A positive prize to a single win (or equivalently a battle prize) can be optimal in dynamic multi-battle contests between two same individuals mainly because it functions to mitigate the discouragement effect in such contests. A zero battle turns out to be optimal in dynamic contests between two teams since there exists no such mitigation effect in such contests as the discouragement effect simply does not exist at all in such setting.

In this paper, we focus on sequential-play three-battle contests. The insight obtained extends to the design of simultaneous-play multi-battle contests. One can check that a winner-take-all best-of-three remains to be optimal for simultaneous-play three-battle team contests, where the discouragement effect is not a concern.\textsuperscript{17} For the same reason, one can reasonably expect that a winner-take-all best-of-three contest be optimal when the battles are simultaneously played between two individuals.\textsuperscript{18}

Our findings provide a rational from a perspective of effort elicitation for the commonly adopted winner-take-all prize allocation in dynamic multi-battle contests, as well as the practice of setting intermediate prizes in many occasions in a single, integrated model. Nevertheless, our analysis sets an upper bound on the maximal prize for the player with a single win. Its optimal level should never go beyond one-third of the total prize budget. In other words, the optimal prize allocation rule should in principle give additional award to the grand winner of the whole contest.

\textsuperscript{17} The symmetric equilibrium where players adopt the same strategy is easy to characterize for all discriminatory power $r > 0$.

\textsuperscript{18} The technical difficulties in analysing the simultaneous-play three-battle contests lie in the equilibrium characterizations for the contests between two same players. We list two relevant studies that adopt this setting in the following: Szentes and Rosenthal(2003a) provides a equilibrium construction for all-pay case under a class of prize allocations that includes winner-take-all; Given winner-take-all prize structure, Klumpp and Polborn (2006) analyses the equilibrium by modelling each battle as Tullock contest with discriminatory power $r \in (0, 1]$.
Appendix

Proof of Lemma 4

**Proof.** Note that \( v(1) - v(0) = \frac{w_A}{2} + \frac{w_B}{2} - \frac{w_A - w_B}{r} \), \( v(2) - v(1) = \frac{w_A}{2} + \frac{w_B}{r} \) and \( \frac{d}{dv}(w_B) = \frac{d}{dv}(w_A) = 1 \).

We then can verify that \( \frac{d}{dv}(w_B) = 0 \) and \( \frac{d}{dv}(w_A) = 1 \). Thus

\[
\frac{d}{dv}(0) \left( \frac{TE_1}{r} \right) = \left( 1 - 2\frac{w_A}{w_A + w_B} \right) \frac{d}{dv}(0) \left( \frac{w_A^2}{w_A + w_B} \right) \left[ (1 + \frac{r^2}{4})(v(1) - v(0)) + (1 - \frac{r^2}{4})(v(2) - v(1)) \right] + (1 - \frac{r^2}{4})(v(1) - v(0)) + \frac{w_A}{w_A + w_B} \frac{d}{dv}(0) \left( \frac{w_A^2}{w_A + w_B} \right) (1 + \frac{r^2}{4}) (-1) + \frac{w_A}{w_A + w_B} \left( v(1) - v(0) \right) + \frac{w_A}{w_A + w_B} (-1)
\]

\[
= \frac{1}{(1 + \eta^r)^3} \left[ \frac{1}{4} \eta^{-1} \left[ (r + 2) + (r - 2) \eta \right] \left[ (1 - \frac{r}{2}) (r + 1 + (r - 1) \eta) \right] - (1 + \eta^r) \left[ 1 + 2 + \frac{2}{4} \eta^r \right] \right],
\]

where \( \eta = \frac{w_B}{w_A} \). Note that \( \frac{1 - \frac{r}{2}}{2 + \frac{r}{4}} \leq \eta \leq \frac{1 + \frac{r}{2}}{2 + \frac{r}{4}} \).

We now prove \( \frac{d}{dv_1}TE_1 < 0 \).

**Case 1:** \( w_A \geq w_B \). In this case, we see \( 1 \geq \eta \) and \( \frac{1 - \frac{r}{2}}{2 + \frac{r}{4}} \geq \eta \geq \frac{1 + \frac{r}{4}}{4 + \frac{r}{4}} \geq \frac{1}{4} \).

Notice that \( \xi = \left[ (1 - \frac{r}{2})(-r - 1 + (r - 1) \eta) \right] + (1 + \frac{r}{2})(-r - 1 + (r + 1) \eta) \) increases in \( \eta \in \left[ \frac{1}{4}, 1 \right] \) for \( r \leq r \in (1, 1.2) \). Thus \( \frac{d}{dv_1} \geq (1 - \frac{r}{2})(r - 1) \eta \) and \( \varphi_1 = (1 - \frac{r}{2})(r - 1) \eta \) increases in \( \eta \in \left[ r - 1, r + 1 \right] \).

When \( r \leq 1 \), we have \( (1 - \frac{r}{2})(r - 1) \eta \geq 1 \) and \( (1 + \frac{r}{2})(1 + r) \eta \) increases in \( \eta \in \left[ 1, \frac{1}{4} \right] \).

When \( r > 1 \), we have \( (1 - \frac{r}{2})(r - 1) \eta \geq 1 \) and \( (1 + \frac{r}{2})(1 + r) \eta \) increases in \( \eta \in \left[ \frac{1}{4}, 1 \right] \).

Thus \( \varphi_1 \geq 0 \) clearly. Thus \( \frac{d}{dv_1} \geq 0 \).

Therefore, we have

\[
(1 + \eta^r)^3 \frac{d}{dv_1}(0) \left( \frac{TE_1}{r} \right) \leq \frac{1}{4} \eta^r \left[ (r + 2) + (r - 2) \eta \right] \left[ (1 - \frac{r}{2})(-2) + (1 + \frac{r}{2})(2) \right] - (1 + \eta^r) \left[ 1 + 2 + \frac{2}{4} \eta^r \right] \leq \frac{1}{4} \eta^r \left[ (r + 2) + (r - 2) \eta \right] - (1 + \eta^r) \left[ 1 + 2 + \frac{2}{4} \eta^r \right],
\]

It suffices to show that \( \frac{r}{2} < \left( \frac{(1 + \eta^r)(1 + 2 + \frac{2}{4})}{\eta^r + (r + 2) + (r - 2) \eta} \right) = \frac{(1 + \eta^r)(1 + 2 + \frac{2}{4})}{\eta^r + (r + 2) + (r - 2) \eta} \) for all \( \frac{1}{4} \leq \eta \leq 1 \).

Note that \( (\eta^1 - r + \eta) [1 + 2 + \frac{2}{4}] \) is increasing in \( \eta \) for \( \frac{1}{4} \leq \eta \leq 1 \) and \( 0 \leq r \leq \frac{1}{4} \). The
monotonicity of \((\eta^{-r} + \eta)[1 + (2 + \frac{r^2}{4})\eta^r]\) is clear when \(r \leq 1\). When \(r \in [1, \overline{r}]\), the first order derivative of \((\eta^{-r} + \eta)[1 + (2 + \frac{r^2}{4})\eta^r]\) is \((1 - r)\eta^{-r} + 1 + (2 + \frac{r^2}{4})(1 + (r + 1)\eta^r) = 1 + (2 + \frac{r^2}{4}) - \frac{1}{\eta^{-r}} + (2 + \frac{r^2}{4})(r + 1)\eta^r \geq 3\frac{1}{4} - \frac{0.2}{(2 + \frac{r^2}{4})} + (2 + \frac{1}{4})(2 + 1)(\frac{1}{4})^{1.2} > 0\). Moreover, \(\frac{r}{2} < ((\frac{1}{4})^{-r} + \frac{1}{4})^{\frac{1 + (2 + \frac{r^2}{4})(\frac{1}{4})^r}{(r+2)(r-2)(\frac{1}{4})}}\) holds for \(0 \leq r \leq \overline{r}\). \(\frac{r}{2} < ((\frac{1}{4})^{-r} + \frac{1}{4})^{\frac{1 + (2 + \frac{r^2}{4})(\frac{1}{4})^r}{(r+2)(r-2)(\frac{1}{4})}}\) is equivalent to \(\frac{r}{2} < ((1 + 4^r)(1 + (2 + \frac{r^2}{4})(\frac{1}{4})^r)\), which is equivalent to \(1 + 4^r + \frac{2}{(r+2)(r-2)(\frac{1}{4})} + 1 - \frac{5}{2}r^2 - 3r > 0\). Let \(\zeta = 1 + 4^r + \frac{2}{(r+2)(r-2)(\frac{1}{4})} + 1 - \frac{5}{2}r^2 - 3r\). We want to show \(\zeta(r) > 0\) for \(0 \leq r \leq \overline{r}\). \(\zeta'(r) = 4^r \ln 4 - \frac{2}{(4^r)^2} \ln 4 + \frac{r}{2} \ln 4 - \frac{r^2}{4} \ln 4 = 4^r \ln 4 - 5r - 3\). Consider the three components \(4^r \ln 4 - 5r - 2\), \(\frac{r}{2} \ln 4 - 1\), and \(-\frac{r^2}{4} \ln 4\). Clearly the last two components are negative. The first component is maximized when \(r = \frac{\ln 5 - 2 \ln (\ln 4)}{\ln 4} \approx 0.3\). Thus the first component is also negative for all relevant \(r \leq 1.2\). We thus know that \(\zeta\) is minimized when \(r = 1.2\). Note \(\zeta(1.2) = 1 + 5.28 + \frac{2}{5.28} + 1.44\ln 4 + 1 - \frac{5}{2} \cdot 1.44 - 3 \cdot 1.2 > 0\). Therefore, \(\zeta(r) > 0\) for \(0 \leq r \leq \overline{r}\).

We thus have \(\frac{d}{d\ln(0)} TE_1 < 0\) in this case.

Case 2: \(w_A \leq w_B\). In this case, we have \(1 \leq \eta \) and \(\eta \leq \frac{\ln 2 + \frac{r}{2}}{\ln 2 - \frac{r}{2}} \leq \frac{\ln 2 + 1.2}{\ln 2 - 1.2} \leq 4\).

Note that \((1 - \frac{r}{2})(-r - 1 + (r + 1)\eta^r) + (1 + \frac{r}{2})\eta(-r + 1 + (r + 1)\eta^r) \geq [(1 - \frac{r}{2})(-r - 1 + (r - 1)\eta^r) + (1 + \frac{r}{2})(-r + 1 + (r + 1)\eta^r)] = 3(\eta^{-r} - 1) > 0\). When \([r + 2] + (r - 2)\eta \leq 0\), it is clear \(\frac{d}{d\ln(0)} TE_1 < 0\) holds. If \([r + 2] + (r - 2)\eta > 0\), it suffices to show that \([(1 - \frac{r}{2})(-r - 1 + (r - 1)\eta^r) + (1 + \frac{r}{2})\eta(-r + 1 + (r + 1)\eta^r)] < \frac{2}{\eta}(\eta^r - \eta)[1 + (2 + \frac{r^2}{4})\eta^r]\) for all \(1 \leq \eta \leq 4\) and \(0 \leq r \leq \overline{r}\), that is, \((1 + \frac{r}{2})\eta(-r + 1 + (r + 1)\eta^r) < \frac{2}{\eta}(\eta^r - \eta)[1 + (2 + \frac{r^2}{4})\eta^r] + (1 - \frac{r}{2})(r + 1 + (1 - r)\eta^r)\).

Note \((1 - \frac{r}{2})(r + 1 + (1 - r)\eta^r) \geq 0\) for all \(1 \leq \eta \leq 4\) and \(r \leq 1.2\). This is clear when \(r \leq 1\). When \(r \in [1, 1.2]\), we have \((r + 1 + (1 - r)\eta^r) \geq 2 - 0.2 \cdot 1.2 > 0\). Thus, we just need to show that \((1 + \frac{r}{2})\eta(-r + 1 + (r + 1)\eta^r) < \frac{2}{\eta}(\eta^r - \eta)[1 + (2 + \frac{r^2}{4})\eta^r]\), which is equivalent to \((1 + \frac{r}{2})(-r + 1 + (r + 1)\eta^r) < \frac{2}{\eta}(\eta^{-r} + 1)[1 + (2 + \frac{r^2}{4})\eta^r]\), i.e. \(1 - \frac{r}{2} - \frac{r^2}{2} + (1 + \frac{r}{2} + \frac{r^2}{2})\eta^r < \frac{2}{\eta}[(2 + \frac{r^2}{4})\eta^r + 3 + \frac{r^2}{4}] + \frac{r}{2}\eta^{-r}\). This holds because \(\frac{r}{2} + \eta + \frac{r^2}{2} - 1 + (\frac{r}{2} - r - \frac{r^2}{2} - 1)\eta^r > 0\), which is implied by \(\frac{r}{2} + (4 - r - \frac{r^2}{2} - 1) = \frac{1}{2}(4 - r^2 - \frac{r^2}{2}) > 0\) and \(\frac{r}{2} + \eta + \frac{r^2}{2} - 1 = \frac{1}{2}(6 + r^2 + \frac{r^2}{2} - r) > 0\) when \(1 \leq \eta \leq 4\) and \(0 \leq r \leq \overline{r} < 1.2\).

Hence, \(\frac{d}{d\ln(0)} TE_1 < 0\) holds for \(0 \leq r \leq \overline{r} < 1.2\) in this case.

Combining the two cases, we conclude that \(\frac{d}{d\ln(0)} TE_1 < 0\). □

**Proof of Property 2**

**Proof.** \(D(\frac{\ln 2 - r}{\ln 2 + r}, r)\) can be simplified as \(\frac{1}{2} r - \frac{r^2}{2 + r} (4(\frac{2 - r}{2 + r})^{2r} + (5r^2 + 12)(\frac{2 - r}{2 + r})^y - 11r^2 + 8)\). Property 2 can be verified by using standard tool such as Mathematica. And one can verify that \(D(\frac{\ln 2 - r}{\ln 2 + r}, r)|_{r = \overline{r}} = \frac{1}{2} r - \frac{r^2}{2 + r} (4(\frac{2 - r}{2 + r})^{2r} + (5r^2 + 12)(\frac{2 - r}{2 + r})^y - 11r^2 + 8) \mid_{r = \overline{r}} = \frac{1}{2} r - \frac{r^2}{2 + r} (5r^2 - 12r + 4) \mid_{r = \overline{r}} < 0\) as \(\overline{r} \in (1, 1.2)\), so that \(D(\frac{\ln 2 - r}{\ln 2 + r}, r)\) has a unique root \(r\) on \([0, 2]\). This generalized property will be further used in Section 3.2 when we talk about the case of \(r \in [\overline{r}, 2]\). □
Proof of Property 3

Proof. We prove that $D(\eta, r)$ is increasing in $\eta$ when $\eta \geq \frac{1 - \frac{r}{4}}{\frac{2}{\frac{3}{2} + \frac{1}{4}}}$ for each $r \in (0, \tau] \subset (0, 1.2]$. Note that we $r \in (0, \tau] \subset (0, 1.2]$, and $\eta \geq \frac{1 - \frac{r}{2}}{\frac{2}{\frac{3}{2} + \frac{1}{4}}} > \frac{1}{3} \Rightarrow |r - 1| = \frac{1}{4}$. The derivative of $D(\eta, r)$ wrt. $\eta$ is given by

\[
\begin{align*}
(\frac{3r}{2} + 1)(1 - \frac{r}{2})(r - 1 + (r - 1)(r + 1)\eta^r) + (1 + \frac{r}{2})(-2r + 2 + (r + 1)(r + 2)\eta^r)\eta^r \\
+ (\frac{3r}{2} - 1)(1 - \frac{r}{2})(r - 1)\eta^{r-1} + (1 + \frac{r}{2})(-r + 1 + (r + 1)^2\eta^r)] + (2\eta(1 + \eta^r)(2 - \frac{3r^2}{4} + \eta^r))' \\
\geq (\frac{3r}{2} + 1)(1 - \frac{r}{2})(r - 1 + (r - 1)(r + 1)\eta^r) + (1 - \frac{r}{2})(-2r + 2 + (r + 1)(r + 2)\eta^r)] \\
+ (\frac{3r}{2} - 1)(1 - \frac{r}{2})(r - 1)\eta^{r-1} + (1 + \frac{r}{2})(-r + 1 + (r + 1)^2\eta^r)] + (2\eta(1 + \eta^r)(2 - \frac{3r^2}{4} + \eta^r))' \\
= [-4r^2 + 3\frac{r^3}{2} + (r - 3r^2 + \frac{11}{4}r^3 - \frac{3}{4}r^4)\eta^{-1} + (3r + 6r^2 + \frac{9}{4}r^3 - \frac{3}{4}r^4)]
\end{align*}
\]

(3)

\[
\begin{align*}
&+ [4 - \frac{3r^2}{2} + (6 + 6r - \frac{3r^2}{2} - \frac{3r^3}{2})\eta^r + (2 + 4r)\eta^{2r}] \\
&= (4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (r - 3r^2 + \frac{11}{4}r^3 - \frac{3}{4}r^4)\eta^{-1} + (6 + 9r + \frac{9}{2}r^2 + \frac{3}{4}r^3 - \frac{3}{4}r^4)\eta^r + (2 + 4r)\eta^{2r} \\
&> (4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (r - 3r^2 + \frac{11}{4}r^3 - \frac{3}{4}r^4)\eta^{-1} + (6 + 9r + \frac{9}{2}r^2 + \frac{3}{4}r^3 - \frac{3}{4}r^4)\eta^r - \frac{1}{4}\eta^{-1} \\
&= (4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (\frac{3}{2} + \frac{13}{4}r^2 - \frac{15}{8}r^2 + \frac{47}{16}r^3 - \frac{15}{16}r^4)\eta^{-1} \\
&\geq (4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (\frac{3}{2} + \frac{13}{4}r^2 - \frac{15}{8}r^2 + \frac{47}{16}r^3 - \frac{15}{16}r^4)\eta^{-1} \\
&= \frac{41}{8} + \frac{39}{16}r - \frac{221}{32}r^2 + \frac{237}{64}r^3 - \frac{45}{64}r^4 \\
&\geq 0.
\end{align*}
\]

Proof of Theorem 1

Proof. By the Lemmas 2 and 4, to find the optimal prize allocation for a given $r \in (0, \tau]$, we only need to identify the optimal $v(2) \in [\frac{1}{2}, 1]$ by maximizing the $TE_1(v(0) = 0, v(2))$. Differentiating ($\frac{TE_1(v(0)=0,v(2))}{r}$) with respect to $v(2)$ gives

\[
\begin{align*}
\frac{d}{dv(2)}(TE_1(v(0) = 0, v(2))) \\
= (1 - 2\frac{w_A^r}{w_A^r + w_B^r})\frac{d}{dv(2)}(\frac{w_A^r}{w_A^r + w_B^r})[(1 + \frac{r^2}{4})v(1) + (1 - \frac{r^2}{4})(v(2) - v(1))] \\
+ \frac{w_A^r}{w_A^r + w_B^r} - \frac{w_B^r}{w_A^r + w_B^r}[(1 + \frac{r^2}{4})(-1) + (1 - \frac{r^2}{4})2] + 1 + \frac{d}{dv(2)}\frac{w_A^r}{w_A^r + w_B^r}v(1) + \frac{w_A^r}{w_A^r + w_B^r}(-1)
\end{align*}
\]

34
Therefore, after substitution and rearrangement, we have

\[
\left[1 + \frac{r^2}{4}\right]v(1) + \left(1 - \frac{r^2}{4}\right)(v(2) - v(1)) - \frac{d}{dv(2)}\left(\frac{w_B^r}{w_A + w_B^r}\right) \\
+ \frac{w_A^r}{w_A^r + w_B^r} \frac{w_B^r}{w_A + w_B^r} \left(1 - \frac{3r^2}{4}\right) + 1 + \frac{d}{dv(2)}\left(\frac{w_B^r}{w_A^r + w_B^r}\right)(v(1) - \frac{w_A^r}{w_A + w_B^r}).
\]

Suppose that \(w_B > w_A\) holds, i.e. \(v(2) \in \left[\frac{1}{2}, \frac{3}{2}\right)\). Note that the monotonicity of prizes implies positive \(w_A\) and \(w_B\): \(w_A = \frac{v}{4}(v(2) - v(1)) + \frac{v}{2}(v(2) - v(0)) - \frac{v}{4}(v(1) - v(0))\); and \(w_B = \frac{v}{4}(v(1) - v(0)) + \frac{v}{2}(v(2) - v(0)) - \frac{v}{2}(v(2) - v(1))\).

Then, we have

\[
I = \frac{w_B}{w_A + w_B^r}\left[1 + \frac{r^2}{4}\right]v(1) + \left(1 - \frac{r^2}{4}\right)(v(2) - v(1)) - \frac{d}{dv(2)}\left(\frac{w_B^r}{w_A + w_B^r}\right) \geq 0 \text{ because } \frac{d}{dv(2)}\left(\frac{w_B^r}{w_A + w_B^r}\right) \geq 0, \quad \text{II} = 1, \quad \text{III} = 1, \quad \text{IV} = \frac{d}{dv(2)}\left(\frac{w_B^r}{w_A + w_B^r}\right)v(1) \geq 0, \text{ and } V = \frac{w_B^r}{w_A + w_B^r} \leq \frac{1}{2}
\]
because \(w_B > w_A\). Thus, \(\frac{d}{dv(2)}\left(\frac{TE_1(v(0)=0,v(2))}{r}\right) > 0\) when \(v(2) \in \left[\frac{1}{2}, \frac{3}{2}\right)\). As a result, \(\frac{TE_1(v(0)=0,v(2))}{r} < \frac{TE_1(v(0)=0,v(2))}{2}\) for any \(v(2) \in \left[0, \frac{3}{2}\right)\).

We only need to consider \(\frac{d}{dv(2)}\left(\frac{TE_1(v(0)=0,v(2))}{r}\right)\) for the remaining case where \(w_B \leq w_A\) holds, i.e. \(v(2) \in \left[\frac{3}{2}, 1\right]\) to pin down the optimal \(v(2)\). Recall \(\eta = \frac{w_B}{w_A}\) can be viewed as a function of \(v(0), v(2)\) and \(r\). Moreover, for a given \(r \in (0,1,2]\), \(\eta(v(0) = 0, v(2), r)\) decreases with \(v(2) \in \left[\frac{3}{2}, 1\right]\), which yields a lower bound \(\frac{1}{2} + \frac{r}{r + 1}\) for \(\eta(v(0) = 0, v(2), r)\) for \(v(2) \in \left[\frac{3}{2}, 1\right]\), given that \(r \in (0,1,2]\).

Similar to \(\frac{d}{dv(2)}\left(\frac{w_B^r}{w_A + w_B^r}\right)\) as in the the proof of Lemma 4, one can verify that

\[
\frac{d}{dv(2)}\left(\frac{w_B^r}{w_A + w_B^r}\right) = -\frac{r}{[1 + \frac{1}{r}][1 + \frac{1}{r}]}\left[\frac{w_A^r}{w_A + w_B^r} - \frac{3r}{4}(w_A + w_B^r)\right].
\]

Recall \(v(1) - v(0) = \frac{w_A + w_B}{2} - \frac{w_A - w_B}{r}\) and \(v(2) - v(1) = \frac{w_A + w_B}{2} + \frac{w_A - w_B}{r}\). Thus

\[
\frac{d}{dv(2)}\left(\frac{TE_1(v(0)=0,v(2))}{r}\right) = \frac{1}{[1 + \frac{1}{r}][1 + \frac{1}{r}]}\left[\frac{3r^2}{4} - \frac{3r^2}{4} + \frac{3r^2}{4} + \eta r\right]
\]

After substitution and rearrangement, we have

\[
\frac{d}{dv(2)}\left(\frac{TE_1(v(0)=0,v(2))}{r}\right) = \frac{1}{[1 + \frac{1}{r}][1 + \frac{1}{r}]}D(\eta, r).
\]

Therefore, \(\text{sign}(\frac{d}{dv(2)}\left(\frac{TE_1(v(0)=0,v(2))}{r}\right)) = \text{sign}(D(\eta, r))\) whenever \(\eta\) is positive. Recall \(\eta(v(0) = 0, v(2), r) = \frac{w_B}{w_A} = \frac{1}{2} + \frac{r}{2}\left[1 - (\frac{3}{2} - 1) + (\frac{3}{2} - 1)\left(\frac{1 - r^2}{4}\right)\right] - \frac{2}{4} + \frac{1}{r}\) when \(r \leq 2\). Thus \(\frac{1}{2} + \frac{r}{2} \geq \eta(v(0) = 0, v(2), r) \geq \frac{1}{2} - \frac{r}{2} + \frac{1}{r}\).
Note $D(\eta(v(0) = 0, v(2), r), r) > 0, \forall v(2) \in [\frac{1}{r}, 1], r \in (0, \bar{r}]$ by Properties 2 and 3. We thus have the optimal $v(2) = 1$ when $r \in (0, \bar{r}]$.

By the Property 2 of $D(\frac{1}{2} - \frac{r}{2} + \frac{1}{r}, r)$, we know $D(\eta(v(0) = 0, v(2) = 1, r), r) = D(\frac{1}{2} - \frac{r}{2} + \frac{1}{r}, r) < 0, \forall r \in (\bar{r}, \bar{r}]$. Moreover, $D(\eta(v(0) = 0, v(2) = \frac{2}{3}, r), r) = 3r^2 + 12 > 0$. By Property 3, let $v^*(2) \in (\frac{2}{3}, 1)$ to be the unique solution of $D(\eta(v(0) = 0, v(2), r), r) = 0, \forall r \in (\bar{r}, \bar{r}]$. We thus have the optimal $v(2) = v^*(2)$ when $r \in (\bar{r}, \bar{r}]$.

**Proof of Lemma 5**

**Proof.** By Lemma 3 and the two formulas $v(1) - v(0) = \frac{w_A + w_B}{2} - \frac{w_A - w_B}{r}$, $v(2) - v(1) = \frac{w_A + w_B}{2} + \frac{w_A - w_B}{r}$, we have

$$TE_1 = \frac{rw_A w_B}{w_A w_B + w_B w_B + w_A w_B} [(1 + \frac{r^2}{4})(v(1) - v(0)) + (1 - \frac{r^2}{4})(v(2) - v(1))] + \frac{r}{2}(v(2) - v(1)) + r(v(1) - v(0)) \frac{w_A}{w_A + w_B},$$

for prize structures in $\mathcal{V}_0$ when $r \in (\bar{r}, 2]$.

Recall $\bar{r} \in (1, 1.2)$. We first look at the case where $r \in (1.3, 2]$. By assumption, any prize structure $\{v(0), v(1), v(2), v(3)\} \in \mathcal{V}_0$ satisfies $w_B \geq w_A$, so that $v(1) - v(0) \geq v(2) - v(1)$ which is equivalent to $v(2) \in [\frac{1}{2}, \frac{2-v(0)}{3}]$.

Construct a new prize structure $\{\bar{v}(0), \bar{v}(1), \bar{v}(2), \bar{v}(3)\} \in \mathcal{V}_1$ such that $\bar{v}(2) - \bar{v}(1) = v(1) - v(0)$ and $\bar{v}(1) - \bar{v}(0) = v(2) - v(1)$. Specifically, let $\bar{v}(2) = \frac{v(1) - v(0) + 1}{2}$, $\bar{v}(1) = 1 - \bar{v}(2)$, $\bar{v}(0) = \bar{v}(1) - (v(2) - v(1))$ and $\bar{v}(3) = 1 - \bar{v}(0)$. One can verify the new structure is eligible, and we have $\bar{w}_A = (\frac{1}{2} - \frac{r}{4})(\bar{v}(1) - \bar{v}(0)) + (\frac{1}{2} + \frac{r}{4})(\bar{v}(2) - \bar{v}(1)) = (\frac{1}{2} - \frac{r}{4})(v(2) - v(1)) + (\frac{1}{2} + \frac{r}{4})(v(1) - v(0)) = w_B$ and $\bar{w}_B = w_A$. The corresponding total effort under the new prize structure is

$$\bar{TE}_1 = \frac{\bar{w}_A \bar{w}_B}{w_A w_B + w_B w_B + w_A w_B} [(1 + \frac{r^2}{4})(\bar{v}(1) - \bar{v}(0)) + (1 - \frac{r^2}{4})(\bar{v}(2) - \bar{v}(1))] + \frac{r}{2}(\bar{v}(2) - \bar{v}(1)) + r(\bar{v}(1) - \bar{v}(0)) \frac{\bar{w}_A}{\bar{w}_A + \bar{w}_B}$$

$$= \frac{rw_A w_B}{w_A w_B + w_B w_B + w_A w_B} [(1 + \frac{r^2}{4})(v(2) - v(1)) + (1 - \frac{r^2}{4})(v(1) - v(0))] + \frac{r}{2}(v(1) - v(0)) + r(v(2) - v(1)) \frac{w_B}{w_A + w_B}.$$  

Recall $v(1) - v(0) = \frac{w_A + w_B}{2} - \frac{w_A - w_B}{r}$, $v(2) - v(1) = \frac{w_A + w_B}{2} + \frac{w_A - w_B}{r}$. We have that for $w_B > w_A$.
and \( r \in [1.25, 2] \),

\[
\overline{TE}_1 - TE_1 = \frac{r w_A^r w_B^r}{w_A + w_B^r} \left[ \left( -\frac{r^2}{2} \right) (v(1) - v(0)) + \frac{r^2}{2} (v(2) - v(1)) \right] + \frac{r}{2} \left[ (v(1) - v(0)) - (v(2) - v(1)) \right] \\
+ r \left[ (v(2) - v(1)) \frac{w_B^r}{w_A^r + w_B^r} - (v(1) - v(0)) \frac{w_A^r}{w_A^r + w_B^r} \right] \\
\geq \frac{r^2}{4} \left( v(2) - v(1) \right) - \frac{r}{2} \left( v(1) - v(0) \right) + \frac{r}{2} \left[ (v(1) - v(0)) - (v(2) - v(1)) \right] \\
+ \frac{r}{w_A^r + w_B^r} \left[ w_B^r \left( (1 + \frac{2}{r}) w_A + (1 - \frac{2}{r}) w_B \right) - w_A^r \left( (1 - \frac{2}{r}) w_A + (1 + \frac{2}{r}) w_B \right) \right] \\
\geq \frac{r}{w_A^r + w_B^r} \left[ \frac{r}{2} \left( (1 + \frac{2}{r}) w_A w_B (w_B^{r-1} - w_A^{r-1}) + (1 - \frac{2}{r}) (w_B^{r+1} - w_A^{r+1}) \right) \right] \\
= \frac{r}{2} \frac{w_A^{r+1}}{w_A^r + w_B^r} \left[ (2 + r) \left( \frac{w_B}{w_A} \right)^r - 4 \frac{w_B}{w_A} + (2 - r) \right] \\
> 0.
\]

When \( \frac{w_B}{w_A} \geq 1 \), \((2 + r)\left(\frac{w_B}{w_A}\right)^r - 4 \frac{w_B}{w_A} + (2 - r)\) strictly increases with \( \frac{w_B}{w_A} \) when \( r \geq 1.25 \). When \( \frac{w_B}{w_A} = 1 \), \((2 + r)\left(\frac{w_B}{w_A}\right)^r - 4 \frac{w_B}{w_A} + (2 - r) = 0 \).

We now look at the case where \( r \in (\tau, 1.25] \). For this case, we first show \( TE_1 \) decreases with \( v(0) \) when \( \frac{w_B}{w_A} \geq 1 \); second we show that \( TE_1(v(0) = 0, v(2)) \) where \( v(2) \in \left[ \frac{1}{2}, \frac{3}{4} \right] \) is strictly dominated by \( TE_1(v(0) = 0, v(2)) = 0 \).

In a first step, we show \( TE_1 \) decreases with \( v(0) \) when \( \frac{w_B}{w_A} \geq 1 \) and \( r \in (\tau, 1.25] \). This can be achieved by following the same procedure as in the case 1 in the proof of Lemma 4. To save space, please refer to the proof of Lemma 4. Note we have \( \eta = \frac{w_B}{w_A} \in [1, \frac{3}{1}] \). Recall in the proof of Lemma 4 \( \varphi_2 = (1 + \frac{1}{2}) (1 - r) \) and \( \varphi_1 = \eta^{-1} \left[ (1 - \frac{1}{2}) (r - 1) + (1 + \frac{1}{2}) (1 + r) \eta (r + 1) \right] \). When \( r \in [\tau, 1.25] \), we have \( \varphi_2 \geq -1.625 \cdot 0.25 = -0.40625 \). \( \varphi_1 \geq -1.625 \cdot 0.25 \cdot 1.25 = 1.3204 > 0.40625 \). Thus \( \frac{\text{d} \varphi_1}{\text{d} \eta} \geq 0 \).

Therefore, we have

\[
(1 + \eta^r)^3 \frac{d}{dx}(\frac{TE_1}{r}) \leq \frac{r}{2} \eta^{-r} \left[ (1 + r) - (2 - r) \eta \right] - (1 + \eta^r) [1 + (2 + \frac{r^2}{4}) \eta^r]. \quad (4)
\]

It suffices to show that \( \frac{r}{2} < \frac{(1 + \frac{1}{2}) (1 + (2 + \frac{r^2}{4}) \eta^r)}{\eta^{-r}(1 + (2 + \frac{r^2}{4}) \eta^r)} \) for all \( r \geq 1.25 \). We show that \((1 + r) - (2 - r) \eta \) is increasing in \( \eta \) for \( \frac{1}{2} \leq \eta \leq 1.25 \). The first order derivative of \((1 + r) - (2 - r) \eta \) is \((1 - r) \eta^{-r} + 1 + (2 + \frac{r^2}{4}) (1 + (r + 1) \eta^r) = 1 + (2 + \frac{r^2}{4}) - \frac{1 - r}{\eta^r} + (2 + \frac{r^2}{4}) (r + 1) \eta^r \)

\[
\geq \frac{3}{4} - \frac{0.25}{\eta^{1.25}} + 2 = (2 + \frac{1}{4})(2 + 1)(\frac{3}{4})^{1.25} = 2.766 > 0. \quad \text{Moreover,} \quad \frac{r}{2} < \frac{(1 + \frac{1}{2}) (1 + (2 + \frac{r^2}{4}) \eta^r)}{\eta^{-r}(1 + (2 + \frac{r^2}{4}) \eta^r)}
\]

holds for \( 1.25 \geq r \geq \tau \). \( \frac{r}{2} < \frac{(1 + \frac{1}{2}) (1 + (2 + \frac{r^2}{4}) \eta^r)}{\eta^{-r}(1 + (2 + \frac{r^2}{4}) \eta^r)} \) is equivalent to \( \frac{r}{2} < \frac{(1 + \frac{1}{2}) (1 + (2 + \frac{r^2}{4}) \eta^r)}{\eta^{-r}(1 + (2 + \frac{r^2}{4}) \eta^r)} \), which
is equivalent to $1 + 4^r + \frac{2}{4^r} + \frac{r^2}{4^r} + 1 - \frac{5}{2}r^2 - 3r > 0$. Let $\varsigma = 1 + 4^r + \frac{2}{4^r} + \frac{r^2}{4^r} + 1 - \frac{5}{2}r^2 - 3r$. We want to show $\varsigma(r) > 0$ for $1.25 \geq r \geq \overline{r}$. $\varsigma'(r) = 4^r \ln 4 - \frac{2}{(4^r)^2} \ln 4 + \frac{r}{4^r} - \frac{r^2}{4^r} \ln 4 - 5r - 3$. Consider the three components $4^r \ln 4 - 5r - 2, \frac{r}{4^r} - 1$ and $-\frac{2}{(4^r)^2} \ln 4 - \frac{r^2}{4^r} \ln 4$. Clearly the last two components are negative. The first component is maximized when $r = \frac{\ln 5 - 2 \ln (\ln 4)}{\ln 4} \approx 0.3$. Thus the first component is also negative for all relevant $r \leq 1.25$. We thus know that $\varsigma$ is minimized when $r = 1.25$. Note $\varsigma(1.25) = 1 + 4^{1.25} + \frac{2}{4^{1.25}} + \frac{1.25^2}{4^{1.25}} + 1 - \frac{5}{2} \cdot 1.25^2 - 3 \cdot 1.25 = 0.4178 > 0$. Therefore, $\varsigma(r) > 0$ for $1.25 \geq r \geq \overline{r}$. We thus have $\frac{d}{dv} TE_1 < 0$ in this case, which means that we can focus on prize structures with $v(0) = 0$ to search for the optimum in $V_0$.

In a second step, we show that $TE_1(v(0) = 0, v(2))$ where $v(2) \in [\frac{1}{2}, \frac{2}{3}]$ (i.e. prizes are in $V_0$) is strictly dominated by $TE_1(v(0) = 0, v(2) = \frac{2}{3})$. The proof is identical to the part of the proof of Theorem 1 to show $\frac{d}{dv} (TE_1(v(0) = 0, v(2))) > 0$ when $w_B > w_A$. Note that prize structure with $v(0) = 0, v(2) = \frac{2}{3}$ is a common element of $V_0 \cap V_1$.

**Proof of Lemma 6**

**Proof.** In $V_1$, we have $w_A \geq w_B$ and $r \leq 1 + \left(\frac{w_B}{w_A}\right)^r$, which imply that $v(1) \leq \frac{1 + v(0)}{3}$, $v(1) \geq \frac{-1}{2} + \frac{3}{4} + \frac{(1 + \frac{2}{3})r(1 - 1)^{1.5}}{-1 + \frac{2}{3} + \frac{1}{3} + \frac{r}{3}(r-1)^{1.5}} + \frac{v(0)}{2}$ and $\eta := \frac{w_B}{w_A} \in [(r-1)^{1.5}, 1]$.

Note that $r > \overline{r}$ gives $-\frac{1}{2} + \frac{3}{4} + \frac{(1 + \frac{2}{3})r(1 - 1)^{1.5}}{-1 + \frac{2}{3} + \frac{1}{3} + \frac{r}{3}(r-1)^{1.5}} > 0$ because of the definition of $\overline{r}$, and thus $-\frac{1}{2} + \frac{3}{4} + \frac{(1 + \frac{2}{3})r(1 - 1)^{1.5}}{-1 + \frac{2}{3} + \frac{1}{3} + \frac{r}{3}(r-1)^{1.5}} > 0$. One can easily verify that prize profile with $v(0) = 0, v(1) = \frac{-1}{2} + \frac{3}{4} + \frac{(1 + \frac{2}{3})r(1 - 1)^{1.5}}{-1 + \frac{2}{3} + \frac{1}{3} + \frac{r}{3}(r-1)^{1.5}}$, and $\eta = (r-1)^{1.5}$ belongs to $V_1$.

To facilitate computation, we first come up a different way of writing the $TE_1$.

$$
TE_1 = \frac{rw^r_A}{[w^r_A + w^r_B]^2}[(1 + \frac{r^2}{4})(v(1) - v(0)) + (1 - \frac{r^2}{4})(v(2) - v(1))] \\
+ \frac{r}{2}(v(2) - v(1)) + r(v(1) - v(0)) - \frac{w^r_A}{w^r_A + w^r_B} \\
= \frac{rw^r_A}{[w^r_A + w^r_B]^2}[(1 - \frac{r}{2})w_A + (1 + \frac{r}{2})w_B] + \frac{r}{2}[(1 + \frac{2}{r})w_A + (1 - \frac{2}{r})w_B] \\
+ \frac{r}{2}[(1 - \frac{2}{r})w_A + (1 + \frac{2}{r})w_B] - \frac{w^r_A}{w^r_A + w^r_B} \\
= \frac{r}{[1 + v^r]^2}[(1 - \frac{r}{2}) + (1 + \frac{r}{2})v]w_A + \frac{r}{4}[(1 + \frac{2}{r}) + (1 - \frac{2}{r})]w_A \\
+ \frac{r}{2}[(1 - \frac{2}{r}) + (1 + \frac{2}{r})v]w_A + \frac{1}{1 + v^r} \\
= [(1 - \frac{r}{2})(r-1)v^r - \frac{1}{(v^r + 1)^2} + \frac{1}{2}(r + 1)]w_A + [(1 + \frac{r}{2})(r+1)v^r + \frac{1}{(v^r + 1)^2} + \frac{1}{2}(r - 1)]w_B
$$
Denote $A := [(1 + \frac{r}{2})(r+1)^{v+1} + \frac{1}{2}(\frac{r}{2} - 1)]$ and $B := [(1 - \frac{r}{2})(r-1)^{v-1} + \frac{1}{2}(\frac{r}{2} + 1)]$. Therefore,

$$TE_1 = Bw_A + Aw_B$$
$$= [(\frac{1}{2} - \frac{r}{4})B + (\frac{1}{2} + \frac{r}{4})A][v(1) - v(0)] + [(\frac{1}{2} + \frac{r}{4})B + (\frac{1}{2} - \frac{r}{4})A](1 - 2v(1))$$
$$= [\frac{1}{2}(A + B) + \frac{3r}{4}(A - B)]v(1) - \frac{1}{2}(A + B) + \frac{r}{4}(A - B)v(0)$$
$$+ \frac{1}{2}(A + B) + \frac{r}{4}(B - A)$$

where $A + B = r[(\frac{3v+1}{(v+1)^2} + \frac{1}{2})$ and $A - B = \frac{1}{(v+1)^2}[(2 + r^2)v - 2] - 1.$

When $v(0) = 0$ and $v(1) = -\frac{1}{2} + \frac{r}{4} + (\frac{1}{2} + \frac{r}{4})(r - 1)\frac{1}{2}$, the total effort induced is given by the following:

$$TE_1(v(0), v(1)) = \frac{1}{2}(A + B) - \frac{1}{2} + \frac{3r}{4} + (\frac{1}{2} + \frac{3r}{4})(r - 1)\frac{1}{2}$$
$$+ \frac{r}{4}(A - B)[\frac{-3}{2} + \frac{3r}{4} + (\frac{3}{2} + \frac{3r}{4})(r - 1)\frac{1}{2} - 1]$$
$$= \frac{r}{4}(A + B) - \frac{1}{2} + \frac{3r}{4} + (\frac{1}{2} + \frac{3r}{4})(r - 1)\frac{1}{2}$$
$$+ \frac{r}{4}(A - B)[\frac{-1}{2} + \frac{3r}{4} + (\frac{1}{2} + \frac{3r}{4})(r - 1)\frac{1}{2}]$$
$$= \frac{Ar(r - 1)^{\frac{1}{2}}}{1 + \frac{3r}{4} + (1 + \frac{3r}{4})(r - 1)^{\frac{1}{2}}} + \frac{rB}{1 + \frac{3r}{4} + (1 + \frac{3r}{4})(r - 1)^{\frac{1}{2}}}$$
$$= \frac{r}{1 + \frac{3r}{2} + (1 + \frac{3r}{2})(r - 1)^{\frac{1}{2}}} [A(r - 1)^{\frac{1}{2}} + B]$$
$$= \frac{r}{1 + \frac{3r}{2} + (1 + \frac{3r}{2})(r - 1)^{\frac{1}{2}}} [(\frac{3r}{4} + \frac{1}{2})(r - 1)^{\frac{1}{2}} + \frac{5}{2} - \frac{r - 2}{r}]$$
$$= \frac{r}{2} + \frac{(r - 1)(2 - r)}{-1 + \frac{3r}{2} + (1 + \frac{3r}{2})(r - 1)^{\frac{1}{2}}}.$$

Similarly, prize profile with $v(0) = 0, v(1) = -\frac{1}{2} + \frac{r}{4} + (\frac{1}{2} + \frac{r}{4})(r - 1)^{\frac{1}{2}} + \frac{r}{4}$ and $\eta = \frac{1}{r - 1}$ belongs to $\mathcal{V}_0$. The
total effort induced is

\[ TE_1(v(0)) = 0, v(1) = \frac{(\frac{r}{2} - 1)(r - 1)^{\frac{1}{2}} + \frac{r}{2} + 1}{(\frac{3r}{2} - 1)(r - 1)^{\frac{1}{2}} + \frac{3r}{2} + 1} \]

\[ = \frac{1}{2} (A + B) \frac{r(1 + (r - 1)^{\frac{1}{2}})}{(\frac{3r}{2} - 1)(r - 1)^{\frac{1}{2}} + \frac{3r}{2} + 1} \]
\[ + \frac{r}{4} (A - B) \frac{-2(r - 1)^{\frac{1}{2}} + 2}{(\frac{3r}{2} - 1)(r - 1)^{\frac{1}{2}} + \frac{3r}{2} + 1} \]
\[ = \frac{1}{(\frac{3r}{2} - 1)(r - 1)^{\frac{1}{2}} + \frac{3r}{2} + 1} r[A + (r - 1)^{\frac{1}{2}}B] \]
\[ = \frac{1}{(\frac{3r}{2} - 1)(r - 1)^{\frac{1}{2}} + \frac{3r}{2} + 1} [(2 + r)(r - 1) + \frac{r}{2}(r - 1) + \frac{r}{2}(r^2 + 1)(r - 1)^{\frac{1}{2}}] \]
\[ = \frac{\frac{5}{2} r^2 + \frac{5}{2} - 2 + (\frac{r^2}{2} + \frac{r}{2}) (r - 1)^{\frac{1}{2}}}{(\frac{3r}{2} - 1)(r - 1)^{\frac{1}{2}} + \frac{3r}{2} + 1} \].

Note that \( v(2) = \frac{2r[(r - 1)^{\frac{1}{2}} + 1]}{3r + 3r(r - 1)^{\frac{1}{2}} + 2(r - 1)^{\frac{1}{2}} - 2} \geq \frac{2}{3} \) and \( v(2) = \frac{2r[(r - 1)^{\frac{1}{2}} + 1]}{3r + 3r(r - 1)^{\frac{1}{2}} + 2(r - 1)^{\frac{1}{2}} - 2} \leq \frac{2}{3} \) when \( r \in [\frac{7}{2}, 2] \).

Moreover, \( (v(0) = 0, v(2) = \frac{2r[(r - 1)^{\frac{1}{2}} + 1]}{3r + 3r(r - 1)^{\frac{1}{2}} + 2(r - 1)^{\frac{1}{2}} - 2}) \) is on the boundary separating \( \mathcal{V}_1 \) and \( \mathcal{V}_2 \); and
\[ (v(0) = 0, v(2) = \frac{2r[(r - 1)^{\frac{1}{2}} + 1]}{3r + 3r(r - 1)^{\frac{1}{2}} + 2(r - 1)^{\frac{1}{2}} - 2}) \] is on the boundary separating \( \mathcal{V}_0 \) and \( \mathcal{V}_3 \). ■

**Proof of Lemma 7**

**Proof.** The game is solved by backward induction. Recall \((n_A, n_B)\) denotes the history of the game.

We first consider the third battle, which can be solved in the same way as Lemma 3.

When \((n_A, n_B) = (2, 0)\) or \((0, 2)\)
\[
x_A(2, 0) = x_B(2, 0) = x_A(0, 2) = x_B(0, 2) = \frac{rv_A(2, 0)}{4} = \frac{r}{4}(v(1) - v(0)).
\]

The winning probabilities are
\[ p_A(2, 0) = p_B(2, 0) = p_A(0, 2) = p_B(0, 2) = \frac{1}{2}. \]

When \((n_A, n_B) = (1, 1)\), we have
\[
x_A(1, 1) = x_B(1, 1) = \frac{rv_A(1, 1)}{4} = \frac{r}{4}(v(2) - v(1)).
\]

The winning probabilities are
\[ p_A(2, 0) = p_B(2, 0) = \frac{1}{2}. \]

Next, we look at the second battle. When \((n_A, n_B) = (1, 0)\), recall from the proof of Lemma 3 that
the effective prize spreads are as follows:

\[ v_A(1, 0) = \frac{1}{2}(1 - v(1) - v(0)) + \frac{r}{4}(1 - 3v(1) + v(0)) = w_A, \]

\[ v_B(1, 0) = \frac{1}{2}(1 - v(1) - v(0)) + \frac{r}{4}(-1 + 3v(1) - v(0)) = w_B. \]

\[ v_A(1, 0) \geq v_B(1, 0) \text{ is equivalent to } w_A \geq w_B. \] As the prize profiles are in \( V_2 \), applying Lemma 1(ii) gives the equilibrium effort:

\[ \tilde{x}_A(1, 0) = \left( \frac{1}{r - 1} \right)^\frac{1}{r} (1 - \frac{1}{r}) v_B(1, 0), \]

\[ \tilde{x}_B(1, 0) = \begin{cases} (1 - \frac{1}{r}) v_B(1, 0) & \text{with probability } q = \frac{v_B(1, 0)}{v_A(1, 0)} \left( \frac{1}{r - 1} \right)^\frac{1}{r}, \\ 0 & \text{with probability } 1 - q. \end{cases} \]

The winning probabilities are

\[ p_A(1, 0) = 1 - \left( 1 - \frac{1}{r} \right) q; \quad p_B(1, 0) = \left( 1 - \frac{1}{r} \right) q. \]

Similarly, when \((n_A, n_B) = (0, 1)\), we have

\[ \tilde{x}_A(0, 1) = \tilde{x}_B(1, 0) = \begin{cases} (1 - \frac{1}{r}) v_B(1, 0) & \text{with probability } q = \frac{v_B(1, 0)}{v_A(1, 0)} \left( \frac{1}{r - 1} \right)^\frac{1}{r}, \\ 0 & \text{with probability } 1 - q. \end{cases} \]

\[ \tilde{x}_B(0, 1) = \tilde{x}_A(1, 0) = \left( \frac{1}{r - 1} \right)^\frac{1}{r} (1 - \frac{1}{r}) v_B(1, 0). \]

The winning probabilities are

\[ p_A(0, 1) = \left( 1 - \frac{1}{r} \right) q; \quad p_B(0, 1) = 1 - \left( 1 - \frac{1}{r} \right) q. \]

We have

\[ E[\tilde{x}_A(0, 1)] = E[\tilde{x}_B(1, 0)] = q(1 - \frac{1}{r}) v_B(1, 0) = (1 - \frac{1}{r}) \left( \frac{1}{r - 1} \right)^\frac{1}{r} \frac{v_B^2(1, 0)}{v_A(1, 0)}. \]

Now we come to the first battle. The common prize spread is

\[ v_A(0, 0) = v_B(0, 0) \]

\[ = \{ p_A(1, 0) [p_A(2, 0) v(3) + p_B(2, 0) v(2) - x_A(2, 0)] + p_B(1, 0) [p_A(1, 1) v(2) + p_B(1, 1) v(1) - x_A(1, 1)] - \tilde{x}_A(1, 0) \} \]

\[ - \{ p_A(0, 1) [p_A(1, 1) v(2) + p_B(1, 1) v(1) - x_A(1, 1)] + p_B(0, 1) [p_A(0, 2) v(1) + p_B(0, 2) v(0) - x_A(0, 2)] - E[\tilde{x}_A(0, 1)] \} \]

\[ = [1 - (1 - \frac{1}{r}) q] [v(2) - v(0)] + (1 - \frac{1}{r}) v_B(1, 0) [q - \left( \frac{1}{r - 1} \right)^\frac{1}{r}]. \]
Thus the effort supply is
\[ x_A(0, 0) = x_B(0, 0) = \frac{r}{4} v_A(0, 0), \]
and winning probabilities are
\[ p_A(0, 0) = p_B(0, 0) = \frac{1}{2}. \]

Aggregating over the three battle, we have the total effort:
\[
TE_2 = 2x_A(0, 0) + x_A(1, 0) + E[\tilde{x}_B(1, 0)] \\
+ p_A(1, 0)(x_A(2, 0) + x_B(2, 0)) + p_B(1, 0)(x_A(1, 1) + x_B(1, 1)) \\
= \frac{r}{2}(1 - 2v(0)) + (1 - \frac{1}{r})(\frac{1}{r - 1})^\frac{1}{2}(2 - r)[\frac{r}{2} - (\frac{1}{2} + \frac{r}{4})v(0) + (\frac{1}{2} - \frac{3r}{4})v(2)].
\]

\[
\square
\]

**Proof of Proposition 1**

**Proof.** We want to maximize effort \( TE_2 \) subject to prize profiles are in \( V_2 \). By definition of \( V_2 \), \( w_A \geq w_B \) and \( 1 + (\frac{w_B}{w_A})^r < r \leq 2 \) which can be written as \((r-1)^\frac{1}{r}w_A > w_B \), i.e. \( v(2) > \frac{r[1+(r-1)^\frac{1}{r}]}{[r-1]^\frac{1}{r}(1+\frac{3}{2}r)+\frac{3r}{2}r-1]} \), which is less than 1 when \( r > \overline{r} \). We thus have

for any prize profile in \( V_2 \), the effort induced is converging to but smaller than \( TE_2(v(0) = 0, v(2) = \frac{r[1+(r-1)^\frac{1}{r}]}{[r-1]^\frac{1}{r}(1+\frac{3}{2}r)+\frac{3r}{2}r-1]} \), which is the following.

\[
TE_2(v(0) = 0, v(2) = \frac{r[1+(r-1)^\frac{1}{r}]}{[r-1]^\frac{1}{r}(1+\frac{3}{2}r)+\frac{3r}{2}r-1]} \\
= \frac{r}{2} + (1 - \frac{1}{r})(\frac{1}{r - 1})^\frac{1}{2}(2 - r)[\frac{r}{2} + (\frac{1}{2} - \frac{3r}{4})\frac{r[1+(r-1)^\frac{1}{r}]}{[r-1]^\frac{1}{r}(1+\frac{3}{2}r)+\frac{3r}{2}r-1]} \\
= \frac{r}{2} + (1 - \frac{1}{r})(\frac{1}{r - 1})^\frac{1}{2}(2 - r)\frac{1}{[(r-1)^\frac{1}{r}(1+\frac{3}{2}r)+\frac{3r}{2}r-1]}r[1+(r-1)^\frac{1}{r}](1 + \frac{3r}{2}) + \frac{3r}{2} - 1) \\
- \frac{3r}{4}r[1+(r-1)^\frac{1}{r} + \frac{1}{2}r(1+(r-1)^\frac{1}{r})] \\
\frac{r}{2} + (1 - \frac{1}{r})(\frac{1}{r - 1})^\frac{1}{2}(2 - r)\frac{1}{[(r-1)^\frac{1}{r}(1+\frac{3}{2}r)+\frac{3r}{2}r-1]}r(r-1)^\frac{1}{r} \\
= \frac{r}{2} + \frac{(r-1)(2 - r)}{[(r-1)^\frac{1}{r}(1+\frac{3}{2}r)+\frac{3r}{2}r-1]}.
\]

\[
\square
\]
Proof of Proposition 2

Proof. The third battle can be analyzed identically as in Lemma 7. We now look at the second battle. Recall \((n_A, n_B)\) denotes the history of the contest. When \((n_A, n_B) = (1, 0)\) the effective prize spreads are respectively

\[
v_A(1, 0) = \frac{1}{2}(1 - v(1) - v(0)) + \frac{r}{4}(1 - 3v(1) + v(0)) = w_A,
\]

\[
v_B(1, 0) = \frac{1}{2}(1 - v(1) - v(0)) + \frac{r}{4}(-1 + 3v(1) - v(0)) = w_B.
\]

Then \(v_A(1, 0) \leq v_B(1, 0)\) is equivalent to \(w_A \leq w_B\). As the prize profiles are in \(V_3\), applying Lemma 1(ii) gives the equilibrium effort:

\[
\tilde{x}_A(1, 0) = \begin{cases} 
(1 - \frac{1}{r})v_A(1, 0) & \text{with probability } q = \frac{v_A(1, 0)}{v_B(1, 0)}(\frac{1}{r-1})^{\frac{1}{2}}, \\
0 & \text{with probability } 1 - q.
\end{cases}
\]

\[
\tilde{x}_B(1, 0) = (\frac{1}{r-1})^{\frac{1}{2}}(1 - \frac{1}{r})v_A(1, 0)
\]

The winning probability are

\[
p_A(1, 0) = (1 - \frac{1}{r})q, \ p_B(1, 0) = 1 - (1 - \frac{1}{r})q.
\]

Similarly, when \((n_A, n_B) = (0, 1)\), we have

\[
\tilde{x}_A(0, 1) = \tilde{x}_B(1, 0) = (\frac{1}{r-1})^{\frac{1}{2}}(1 - \frac{1}{r})v_A(1, 0).
\]

\[
\tilde{x}_B(0, 1) = \tilde{x}_A(1, 0) = \begin{cases} 
(1 - \frac{1}{r})v_A(1, 0) & \text{with probability } q = \frac{v_A(1, 0)}{v_B(1, 0)}(\frac{1}{r-1})^{\frac{1}{2}}, \\
0 & \text{with probability } 1 - q.
\end{cases}
\]

The winning probability are

\[
p_A(0, 1) = 1 - (1 - \frac{1}{r})q, \ p_B(0, 1) = (1 - \frac{1}{r})q.
\]

We have

\[
E[\tilde{x}_B(0, 1)] = E\tilde{x}_A(1, 0) = q(1 - \frac{1}{r})v_A(1, 0) = (1 - \frac{1}{r})(\frac{1}{r-1})^{\frac{1}{2}}\frac{v^2_A(1, 0)}{v_B(1, 0)}.
\]
Now we come to the first battle. We pin down the common effective prize spread:

\[ v_A(0, 0) = v_B(0, 0) = \left\{ p_A(1, 0)[p_A(2, 0)v(3) + p_B(2, 0)v(2) - x_A(2, 0)] + p_B(1, 0)[p_A(1, 1)v(2) + p_B(1, 1)v(1) - x_A(1, 1)] - E[\tilde{\pi}_A(1, 0)] \right\} - \left\{ p_A(0, 1)[p_A(1, 0)v(2) + p_B(1, 0)v(1) - x_A(1, 1)] + p_B(0, 1)[p_A(0, 2)v(1) + p_B(0, 2)v(0) - x_A(0, 2)] - \tilde{\pi}_A(0, 1) \right\} \]

Thus we have the effort supply

\[ x_A(0, 0) = x_B(0, 0) = \frac{r}{4}v_A(0, 0), \]

and the winning probabilities

\[ p_A(0, 0) = p_B(0, 0) = \frac{1}{2}. \]

Total effort thus is as follow:

\[
TE_3 = 2x_A(0, 0) + E[\tilde{\pi}_A(1, 0)] + \tilde{\pi}_B(1, 0) = \frac{r}{2}(v(2) - v(1)) + r(1 - \frac{1}{r})q(v(1) - v(0)) + (1 + \frac{r}{2})(1 - \frac{1}{r})\left( \frac{1}{r - 1} \right)^\frac{1}{2}w_A + (1 - \frac{r}{2})(1 - \frac{1}{r})qw_A
\]

\[
= \frac{r}{2}(v(2) - v(1)) + r(1 - \frac{1}{r})\left( \frac{1}{r - 1} \right)^\frac{1}{2}w_A + (1 - \frac{r}{2})(1 - \frac{1}{r})\left[ (1 - \frac{2}{r})w_A + (1 + \frac{2}{r})w_B \right]
\]

\[
+ (1 + \frac{r}{2})(1 - \frac{1}{r})\left( \frac{1}{r - 1} \right)^\frac{1}{2}w_A + (1 - \frac{r}{2})(1 - \frac{1}{r})\left( \frac{1}{r - 1} \right)^\frac{1}{2}w_A w_A w_B
\]

\[
= \frac{r}{2}(2v(2) - 1) + (1 - \frac{1}{r})(\frac{1}{r - 1})^{\frac{1}{2}}(2 + r)w_A
\]

\[
= \frac{r}{2}(2v(2) - 1) + (1 - \frac{1}{r})(\frac{1}{r - 1})^{\frac{1}{2}}(2 + r)[(\frac{1}{2} + \frac{r}{4})(2v(2) - 1) + (\frac{1}{2} - \frac{r}{4})(1 - v(0) - v(2))]
\]

\[
= \frac{r}{2}(2v(2) - 1) + (1 - \frac{1}{r})(\frac{1}{r - 1})^{\frac{1}{2}}(2 + r)[-\frac{r}{2} - (\frac{1}{2} - \frac{r}{4})v(0) + (\frac{1}{2} + \frac{3}{4}r)v(2)].
\]

Hence, \( TE_3 \) is increasing in \( v(2) \) and decreasing in \( v(0) \).

We want to maximize effort \( TE_3 \) subject to prize profiles are in \( V_3 \), i.e. \( w_A \leq w_B \) and \( 1 + \frac{w_A}{w_B}r < r \leq 2 \), as well as the non-negativity and monotonicity of the prizes. Since \( r \in (\tau, 2] \), constraints \( w_A \leq w_B \) and \( 1 + \frac{w_A}{w_B}r < r \leq 2 \) can be written as \( (r - 1)^{\frac{1}{2}}w_B > w_A \), i.e. \( v(2) < \frac{r(1+\tau)^{\frac{1}{2}}}{(r-1)^{\frac{1}{2}}\tau + (\frac{1}{2} + \frac{3}{4}r)}v(0) \). Clearly, \( TE_3 \) is decreasing in \( v(0) \) and increasing in \( v(2) \) when \( r \in (\tau, 2] \).
thus can set \( v(0) = 0 \) and consider simplified constraint \( v(2) < \frac{r(1+(r-1)^{\frac{1}{2}})}{(r-1)^{\frac{1}{2}}(\frac{3r}{2} - 1) + \left(\frac{3r}{2} + 1\right)} \), which is less than 1 when \( r > \tau \). We thus have for any prize profile in \( \mathcal{V}_3 \), the effort induced is converging to but smaller than \( TE_3(v(0) = 0, v(2) = \frac{r(1+(r-1)^{\frac{1}{2}})}{(r-1)^{\frac{1}{2}}(\frac{3r}{2} - 1) + \left(\frac{3r}{2} + 1\right)} \), which is the following.

\[
TE_3(v(0) = 0, v(2) = \frac{r(1+(r-1)^{\frac{1}{2}})}{(r-1)^{\frac{1}{2}}(\frac{3r}{2} - 1) + \left(\frac{3r}{2} + 1\right)} ) = r \left( \frac{1}{(r-1)^{\frac{1}{2}}(\frac{3r}{2} - 1) + \left(\frac{3r}{2} + 1\right)} \right) + (1 - \frac{1}{r}) \left( \frac{1}{(r-1)^{\frac{1}{2}}} \right)^2 (2 + r) [- \frac{r}{2} + \left(\frac{3}{4} + r\right) \frac{1}{(r-1)^{\frac{1}{2}}(\frac{3r}{2} - 1) + \left(\frac{3r}{2} + 1\right)}]
\]

\[
= \frac{1}{(r-1)^{\frac{1}{2}}(\frac{3r}{2} - 1) + \frac{3}{2} r + 1} \{ r^2 (1 + (r-1)^{\frac{1}{2}}) - \frac{r}{2} (r-1)^{\frac{3}{2}} [\frac{3}{2} r - 1] + \frac{3}{2} r + 1 ) + (1 - \frac{1}{r}) \left( \frac{1}{(r-1)^{\frac{1}{2}}} \right)^2 (2 + r) [ - \frac{r}{2} + \left(\frac{3}{4} + r\right) r (1 + (r-1)^{\frac{1}{2}}) ] \}
\]

\[
= \frac{1}{(r-1)^{\frac{1}{2}}(\frac{3r}{2} - 1) + \frac{3}{2} r + 1} \{ r^2 (1 + (r-1)^{\frac{1}{2}}) - \frac{r}{2} (2 + r) [r (1 + (r-1)^{\frac{1}{2}}) - \frac{3}{2} r - 1] - \frac{r}{2} \left(\frac{3}{2} r + 1\right) + (1 - \frac{1}{r}) \left( \frac{1}{(r-1)^{\frac{1}{2}}} \right)^2 (2 + r) [r (1 + (r-1)^{\frac{1}{2}}) - \frac{3}{2} r - 1] \}
\]

\[
= \frac{1}{(r-1)^{\frac{1}{2}}(\frac{3r}{2} - 1) + \frac{3}{2} r + 1} \{ r^2 + r^2 (r-1)^{\frac{1}{2}} - \frac{3}{4} r^2 - \frac{r}{2} (r-1)^{\frac{1}{2}} - \frac{3}{4} r^2 - \frac{r}{2} + (r-1)(2 + r) \}
\]

\[
= \frac{1}{(1 + \frac{3r}{2} + (\frac{3r}{2} - 1)(r-1)^{\frac{1}{2}})}.
\]

**Proof of Theorem 2**

**Proof.** Step 1: we prove \( \frac{d}{dv(0)} \left( \frac{TE_1}{r} \right) < 0 \) when \( r \in (\tau, 2] \) and \( \eta \in [(r-1)^{\frac{1}{2}}, 1] \) as follows. From the proof of Lemma 4, we know that

\[
\frac{d}{dv(0)} \left( \frac{TE_1}{r} \right) = \frac{1}{(1 + \eta r)^3} \left\{ \frac{1}{4} \eta r^{-1} [(r + 2) + (r - 2)\eta] [(1 - \frac{r}{2}) (-r - 1 + (r - 1)\eta)] + \right.
\]

\[
\left. (1 + \frac{r}{2}) (-r + 1 + (r - 1)\eta) - (1 + \eta) [1 + (2 + \frac{r^2}{4}) \eta] \right\}.
\]

Note that \([(1 - \frac{r}{2}) (-r - 1 + (r - 1)\eta)] + (1 + \frac{r}{2}) (-r + 1 + (r - 1)\eta)] \) is increasing in \( \eta \) as its derivative \( (1 - \frac{r}{2}) (-r + 1 + (r - 1)\eta) + (1 + \frac{r}{2}) (-r + 1 + (r - 1)\eta) \) is increasing in \( \eta \) as its derivative \( (1 - \frac{r}{2}) (-r + 1 + (r - 1)\eta) + (1 + \frac{r}{2}) (-r + 1 + (r - 1)\eta) \) is increasing in \( \eta \) as its derivative \( (1 - \frac{r}{2}) (-r + 1 + (r - 1)\eta) + (1 + \frac{r}{2}) (-r + 1 + (r - 1)\eta) \). Thus, it suffices to show that \( \frac{1}{4} \eta r^{-1} [(r + 2) + (r - 2)\eta] [(1 - \frac{r}{2}) (-2) + (1 + \frac{r}{2}) 2] \) \( (1 + \eta) [1 + (2 + \frac{r^2}{4}) \eta] \), that is, \( \frac{r}{2} < \frac{(\eta + 1)(2 + \frac{r^2}{4}) \eta}{(r-2) + (r-2)\eta} \), where \( \frac{(\eta + 1)(2 + \frac{r^2}{4}) \eta}{(r-2) + (r-2)\eta} \) is an increasing function in \( \eta \). Hence, it suffices
to show that \( \frac{x}{Y} < \frac{(n^1 + 2 + \frac{r^2}{2})}{(r + 2) + (r - 2) \eta} \bigg|_{\eta = (r - 1)^{\frac{1}{4}}} \), which holds when \( r \in (\tau, 2] \) and \( \eta \in [(r - 1)^{\frac{1}{4}}, 1] \).

Step 2: From the proof of Theorem 1, we obtain that \( \text{sign}(\frac{d}{dr}(\text{TE}(v(0) = 0, v(2)))) = \text{sign}(D(\eta, r)), \) where 
\[
D(\eta, r) \equiv [(3^2 - 1) + ((3^2 + 1)\eta)(1 - \frac{r}{2})(-r - 1 + (r - 1)\eta^r) + (1 + \frac{r}{2})(-r + 1 + (r + 1)\eta^r)\eta] + 2\eta(1 + \eta^r)(2 - \frac{3r^2}{4} + \eta^r).
\]

In this step, we show that \( D(\eta, r) \) is an increasing function in \( \eta \) when \( \eta \in [(r - 1)^{\frac{1}{7}}, 1] \subset \left[\frac{4 - r}{4 + r}, 1\right] \) for each \( r \in (\tau, 2] \).

Recall that we calculate the derivative of \( D(\eta, r) \) with respect to \( \eta \) in the proof of Theorem 1, which is greater than
\[
(4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (r - 3r^2 + \frac{11}{4}r^3 - \frac{3}{4}r^4)\eta^r - 1 + (6 + 9r + \frac{9}{2}r^2 + \frac{3}{4}r^3 - \frac{3}{4}r^4) \eta^r + (2 + 4r)\eta^{2r} > (4 - \frac{11r^2}{2} + \frac{3r^3}{2}) + (6 + 9r + \frac{9}{2}r^2 + \frac{3}{4}r^3 - \frac{3}{4}r^4)(r - 1)
\]

(8)

> 0.

The first inequality holds as \( (r - 3r^2 + \frac{11}{4}r^3 - \frac{3}{4}r^4) > 0 \) when \( r \in (\tau, 2] \) and \( \eta \in [(r - 1)^{\frac{1}{7}}, 1] \).

The rest proof is same as the one in the Theorem 1, except that we use \( (r - 1)^{\frac{1}{7}} \) as the lower bound of \( \eta \) when \( r \in (\tau, 2] \), rather than \( \frac{4 - r}{4 + r} \) as before. 

**Proof of Lemma 9**

**Proof.** We use backward induction to solve the game. Note Lemma 1(ii) applies to all three battles. The third-battle results for history (2,0), (1,1) and (0,2) remain same as in the proof of Lemma 8. Next, we look at the second battle. The expressions for the prize spreads remain same as in the proof of Lemma 8.

History (1,0): \( \tilde{v}_A(1, 0) \leq \tilde{v}_B(1, 0) \) if and only if \( v(2) - v(1) \leq v(1) - v(0) \), and we are considering \( V_5 \).

Thus the effort supply is

\[
G^A_{(1,0)}(x) = \frac{(v(1) - v(0)) - (v(2) - v(1)) + x}{(v(1) - v(0))} \text{ in } [0, v(2) - v(1)],
\]

\[
G^B_{(1,0)}(x) = \frac{x}{v(2) - v(1)} \text{ in } [0, v(2) - v(1)].
\]

The winning probabilities are

\[
p_A(1, 0) = \frac{1}{2} q, \ \ p_B(1, 0) = 1 - \frac{q}{2},
\]

where \( q = \frac{v(2) - v(1)}{v(1) - v(0)} \) We thus have

\[
E[\tilde{v}_A(1, 0)] = \frac{1}{2}(v(1) - v(0)), \ E[\tilde{v}_B(1, 0)] = \frac{1}{2} \frac{(v(2) - v(1))^2}{(v(1) - v(0))}.
\]

46
History (0,1) is the dual case of history (1,0).
Now we come to the first battle. We first pin down the common effective prize spread as follows:

\[ v_A(0,0) = v_B(0,0) \]
\[ = \{ p_A(1,0)[p_A(2,0)v(3) + p_B(2,0)v(2) - E[\tilde{x}_A(2,0)]] \]
\[ + p_B(1,0)[p_A(1,1)v(2) + p_B(1,1)v(1) - E[\tilde{x}_A(1,1)]] - E[\tilde{x}_A(1,0)] \}
\[ - \{ p_A(0,1)[p_A(1,1)v(2) + p_B(1,1)v(1) - E[\tilde{x}_A(1,1)]] \]
\[ + p_B(0,1)[p_A(0,2)v(1) + p_B(0,2)v(0) - E[\tilde{x}_A(0,2)] - x_A(0,1) \}
\[ = \frac{1}{2}q(v(2) - v(0)) + \left( \frac{1}{2} - \frac{q}{2} \right)(v(2) - v(1)). \]

Thus the effort supply is
\[ G^A_{(0,0)}(x) = G^B_{(0,0)}(x) = \frac{x}{v_A(0,0)} \quad \text{in} \quad [0, v_A(0,0)]. \]

The winning probabilities are
\[ p_A(0,0) = p_B(0,0) = \frac{1}{2}. \]

Total effort thus is as follow:
\[ TE_5 = 2E[\tilde{x}_A(0,0)] + E[\tilde{x}_A(1,0)] + E[\tilde{x}_B(1,0)] \]
\[ + p_A(1,0)(E[\tilde{x}_A(2,0)] + E[\tilde{x}_A(2,0)]) + p_B(1,0)(E[\tilde{x}_A(1,1)] + E[\tilde{x}_A(1,1)]) \]
\[ = v_A(0,0) + \left( \frac{1}{2} + q \right)(v(2) - v(1)) - \frac{q}{2}(v(1) - v(0)) + \left( 1 - \frac{q}{2} \right)(v(2) - v(1)) \]
\[ = 3(v(2) - v(1)). \]

**Proof of Lemma 10**

**Proof.** Note that Lemma 1(i) applies to all three battles when \( r \leq 2 \). Before a battle is fought, the history of past battles, or the state of the contest, is observed by players involved. The history of the contest is denoted \((n_A, n_B)\) where \( n_A \) is the number of wins secured by team \( i = A, B \). We solve the game by backward induction. We first look at the third battle.

History (2,0): the effective prize spreads are

\[ v_A(2,0) = v(3) - v(2) \geq 0, \]
\[ v_B(2,0) = v(1) - v(0) \geq 0. \]

The budget constraint \( v(3) + v(0) = v(2) + v(1) \) implies \( v(3) - v(2) = v(1) - v(0) \), so \( v_A(2,0) = v_B(2,0) \)
By Lemma 1(i), we have effort supply
\[ x_A(2, 0) = \frac{rv_r^r + 1(2, 0)v_B(2, 0)(2, 0)}{rv_A(2, 0)} = \frac{rv_A(2, 0)}{4}, \]
\[ x_B(2, 0) = \frac{rv_r^r + 1(2, 0)v_A(2, 0)(2, 0)}{rv_B(2, 0)} = \frac{rv_B(2, 0)}{4}. \]

The winning probabilities are
\[ p_A(2, 0) = \frac{x_A^r(2, 0)}{x_A^r(2, 0) + x_B^r(2, 0)} = \frac{1}{2}, \]
\[ p_B(2, 0) = \frac{x_B^r(2, 0)}{x_A^r(2, 0) + x_B^r(2, 0)} = \frac{1}{2}. \]

History (0,2) is similar. We now look at history (1,1). For history (1,1), the common prize spread is
\[ v_A(1, 1) = v_B(1, 1) = v(2) - v(1). \]

Thus effort supply is
\[ x_A(1, 1) = x_B(1, 1) = \frac{r}{4}(v(2) - v(1)). \]

The winning probabilities are
\[ p_A(1, 1) = p_B(1, 1) = \frac{1}{2}. \]

We now look at the second battle. The history can be (1,0) or (0,1).

History (1,0): we first pin down the effective prize spreads:
\[ v_A(1, 0) = [p_A(2, 0)v(3) + p_B(2, 0)v(2)] - [p_A(1, 1)v(2) + p_B(1, 1)v(1)] \]
\[ = \frac{1}{2}(v(3) - v(1)) = \frac{1}{2}(v(2) - v(0)), \]
\[ v_B(1, 0) = [p_B(1, 1)v(2) + p_A(1, 1)v(1)] - [p_B(2, 0)v(1) + p_A(2, 0)v(0)] \]
\[ = \frac{1}{2}(v(2) - v(0)). \]

Thus effort supply is
\[ x_A(1, 0) = x_B(1, 0) = \frac{r}{4}v_A(1, 0) = \frac{r}{8}(v(2) - v(0)). \]

The winning probabilities are
\[ p_A(1, 0) = p_B(1, 0) = \frac{1}{2}. \]

History (0,1): This is dual case of history (1,0). We have prize spread
\[ v_A(0, 1) = v_B(0, 1) = \frac{1}{2}(v(2) - v(0)), \]
and effort supply
\[ x_A(0, 1) = x_B(0, 1) = \frac{r}{8}(v(2) - v(0)). \]
The winning probabilities are
\[ p_A(0, 1) = p_B(0, 1) = \frac{1}{2}. \]
Now we come to the first battle. We pin down the common effective prize spread:
\[
v_A(0, 0) = v_B(0, 0) = \{p_A(1, 0)[p_A(2, 0)v(3) + p_B(2, 0)v(2)] + p_B(1, 0)[p_A(1, 1)v(2) + p_B(1, 1)v(1)]
- \{p_A(0, 1)[p_A(1, 1)v(2) + p_B(1, 1)v(1)] + p_B(0, 1)[p_A(0, 2)v(1) + p_B(0, 2)v(0)]\}
= \frac{1}{2}(v(2) - v(0)).
\]
Thus effort supply is
\[ x_A(0, 0) = x_B(0, 0) = \frac{r}{4}v_A(0, 0) = \frac{r}{8}(v(2) - v(0)). \]
The winning probabilities are
\[ p_A(0, 1) = p_B(0, 1) = \frac{1}{2}. \]
Thus, total effort can be calculated as follow:
\[
TE^1_S = 2x_A(0, 0) + [x_A(1, 0) + x_B(1, 0)] + p_A(1, 0)[x_A(2, 0) + x_B(2, 0)] + p_B(1, 0)[x_A(1, 1) + x_B(1, 1)]
= 2 \times [\frac{r}{8}(v(2) - v(0))] + 2 \times [\frac{r}{8}(v(2) - v(0))]
+ \frac{1}{2} \times 2 \times \frac{r}{4}(v(1) - v(0)) + \frac{1}{2} \times 2 \times \frac{r}{4}(v(2) - v(1))
= \frac{3r}{4}(v(2) - v(0)).
\]
\[ \square \]

**Proof of Lemma 11**

**Proof.** Note Lemma 1(iii) applies to each battle. Moreover, Fu, Lu and Pan (2015) reveals that the prize spread is common for the two players in each battle. We solve the game by backward induction. We first look at the third battle.

History (2,0): For history (2,0), we first describe the two players’ effective prize spreads:
\[ v_A(2, 0) = v(3) - v(2) \geq 0, v_B(2, 0) = v(1) - v(0) \geq 0. \]

We have \( v_A(2, 0) = v_B(2, 0) \) follows from the budget constraints.
Thus effort supply is
\[ \tilde{x}_A(2,0) \sim G^A_{(2,0)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)], \]
\[ \tilde{x}_B(2,0) \sim G^B_{(2,0)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)], \]
where \( G^i_{(n_A,n_B)}(\cdot) \) denotes the cumulative distribution function of the player \( i \)'s mixed strategy in equilibrium. The winning probabilities are
\[ p_A(2,0) = p_B(2,0) = \frac{1}{2}. \]
History (0,2) is similar. We now look at history (1,1). For history (1,1), the common effective prize spread is
\[ v_A(1,1) = v_B(1,1) = v(2) - v(1) \geq 0. \]
Thus effort supply is given by
\[ G^A_{(1,1)}(x) = G^B_{(1,1)}(x) = \frac{x}{v(2) - v(1)} \text{ in } [0, v(2) - v(1)]. \]
The winning probabilities are
\[ p_A(1,1) = p_B(1,1) = \frac{1}{2}. \]
History (0,2): For history (0,2), we first describe the two players’ effective prize spreads.
\[ v_A(0,2) = v_B(0,2) = v(1) - v(0) \geq 0 \]
Thus
\[ G^A_{(1,1)}(x) = G^B_{(1,1)}(x) = \frac{x}{v(1) - v(0)} \text{ in } [0, v(1) - v(0)]. \]
Then teams’ winning probability are
\[ p_A(1,1) = p_B(1,1) = \frac{1}{2}. \]
History (1,0): we first pin down the common effective prize spread:
\[ v_B(1,0) = v_A(1,0) = [p_A(2,0)v(3) + p_B(2,0)v(2)] - [p_A(1,1)v(2) + p_B(1,1)v(1)] \]
\[ = \frac{1}{2}(v(3) - v(1)). \]
Thus effort supply is
\[ G^A_{(1,0)}(x) = G^B_{(1,0)}(x) = \frac{2x}{v(2) - v(0)} \text{ in } [0, \frac{1}{2}(v(2) - v(0))]. \]
The winning probabilities are 
\[ p_A(1, 0) = p_B(1, 0) = \frac{1}{2}. \]

History \((0,1)\) is symmetric.

Total effort thus is as follow:

\[
\begin{align*}
TE_S^2 &= 2E[\tilde{x}_A(0, 0)] + (E[\tilde{x}_A(1, 0)] + E[\tilde{x}_B(1, 0)]) \\
&\quad + p_A(1, 0)(E[\tilde{x}_A(2, 0)] + E[\tilde{x}_B(2, 0)]) \\
&\quad + p_B(1, 0)(E[\tilde{x}_A(1, 1)] + E[\tilde{x}_B(1, 1)]) \\
&= 2E[\tilde{x}_A(0, 0)] + 2E[\tilde{x}_A(1, 0)] + E[\tilde{x}_A(2, 0)] + E[\tilde{x}_A(1, 1)] \\
&= \frac{3}{2}(v(2) - v(0)),
\end{align*}
\]

Where

\[
\begin{align*}
E[\tilde{x}_A(0, 0)] &= \int \tilde{x}_A(0, 0)dG_A^{(0,0)}(x) = \frac{1}{4}(v(2) - v(0)), \\
E[\tilde{x}_A(1, 0)] &= \frac{1}{4}(v(2) - v(0)), \\
E[\tilde{x}_A(2, 0)] &= \frac{1}{4}(v(1) - v(0)), \\
E[\tilde{x}_A(1, 1)] &= \frac{1}{4}(v(2) - v(1)).
\end{align*}
\]

References


