Efficient Job Upgrading, Search on the Job and Output Dispersion*

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Abstract

A worker's job can be improved internally through job upgrading or externally through on-the-job search. Incorporating these internal and external job dynamics into a directed search model, I analytically characterize and quantitatively evaluate the socially efficient creation of vacancies, search and job upgrading. The analysis shows that efficient job upgrading continues throughout a worker's career and may be hump shaped over tenure. In contrast, on-the-job search is front-loaded in a worker's career and stops after a finite number of job switches. The dynamic interaction between job upgrading and on-the-job search can generate large frictional dispersion in output among identical workers. The calibrated model yields the mean-min ratio in output as 2.04, which is empirically plausible and much larger than in previous models. Both job upgrading and on-the-job search generate significant dispersion in output, although job upgrading is more potent than on-the-job search. Output dispersion depends critically on the calibrated feature that the marginal cost of a vacancy increases in the job type. If the marginal cost of a vacancy were non-increasing in the job type, the social planner would start all jobs at a high type and leave very little room for job upgrading or on-the-job search.

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1. Introduction

Search frictions generate unemployment. To study this effect, the search literature pioneered by Diamond (1982), Mortensen (1982) and Pissarides (2000) has often assumed that a worker's productivity is fixed over time, with exceptions discussed later. In reality, vacancies differ in the job type and similar workers follow different paths of productivity. A worker's productivity can grow internally in a firm as the job is upgraded or externally as the worker switches to a job of a higher type.[1] This paper examines analytically and quantitatively how job upgrading and on-the-job search interact with each other in the socially efficient allocation.

Internal and external growth of productivity should be studied together, because both can serve as mechanisms to cope with frictions. Anticipating job upgrading and on-the-job search, unemployed workers are willing to accept relatively low jobs, in which case firms are willing to create more vacancies to reduce unemployment. A formal characterization of the efficient allocation with job upgrading and on-the-job search can shed light on a number of questions. How does job mobility in the market affects job upgrading within a firm? Should jobs be upgraded more intensively in the early or the later stage of a worker's career? When jobs can be upgraded, why is it still socially efficient to move workers to other jobs through on-the-job search?

On the quantitative side, it is useful to evaluate the importance of job upgrading and one-the-job search. A particular lens to view this importance is output dispersion among identical workers. Specifically, Hornstein et al. (2011) have measured frictional dispersion by the mean-min ratio of wages induced by search frictions. They concluded that most search models fail to produce significant frictional dispersion. For example, the canonical search model yields the mean-min ratio in wages lower than 1.05, while the empirical value is about 2. Since job upgrading and on-the-job search stretch the two ends of the distribution in output, it is hopeful that they may generate significant output dispersion.

[1] For the evidence on internal labor mobility, see Baker et al. (1994). Lazear and Oyer (2004) provide empirical evidence on both internal and external labor market mobility.

The model economy has identical workers and heterogeneous jobs. Output increases in the job type initially but eventually reaches the maximum at the final job type. The cost of a vacancy is increasing and convex in the job type. Search is directed toward particular job types.[2] After matching with a worker, the firm can upgrade the job. The investment incurred for upgrading jobs is specific to the match and, as such, it is lost when the worker moves to another firm. I characterize the (constrained) efficient allocation chosen by the social planner under the same search frictions as in the market. For workers currently employed at each job type or unemployed, the planner chooses the job type to search and the matching rate at the search target. The specified matching rate is delivered by creating the right number of vacancies at the search target. For a worker who stays at the current job, the planner chooses the rate at which the job is upgraded. The focus on the efficient allocation puts a strong requirement on the model, because not all models of wage dispersion can generate output dispersion in the efficient allocation. For example, Burdett and Mortensen (1998), Burdett and Coles (2003) and Shi (2009) generate dispersion in wages but not in output.

The efficient allocation exhibits the following features. Low type jobs are created to match with unemployed workers. Once employed, a worker's job type starts to be upgraded. At the same time, the worker is allocated to search for a higher job type. If the worker finds a match at the search target, the worker moves to the new job. While job upgrading is continuous, a success in on-the-job search increases the job type by a discrete amount. The job switching rate declines as the worker's job type increases, and reaches zero after a finite number of job-to-job transitions. In contrast, job upgrading continues until the job type reaches the final level at which output ceases to increase.

These results are intuitive. It is socially efficient to create low type jobs for unemployed workers because the marginal cost of vacancies is low for low-type jobs and the option value of unemployment is low. Rather than continuing to search for better jobs, an unemployed worker can take up a job and let the job improve through upgrading and on-the-job search.

[2]For examples of directed search models, see Peters (1991), Moen (1997), Acemoglu and Shimer (1999), Burdett et al. (2001) and Shi (2001, 2009).

In the early part of a worker's employment, job switching is socially efficient despite the existence of job upgrading because a job switch increases the job type more quickly than job upgrading does. As the job type increases, the job switching rate declines because the marginal cost of a vacancy increases in the job type but the marginal gain from a higher job type diminishes in the job type. After a finite number of job switches, the marginal cost of a vacancy exceeds the expected marginal benefit of a vacancy, at which time on-the-job search stops. However, job upgrading continues because the amount of job upgrading can be made sufficiently small.

Job upgrading and on-the-job search are substitutes. Since both increase productivity, it may seem socially efficient to front-load them, i.e., make them happen early in a worker's career. Indeed, on-the-job search is front-loaded. Because the marginal cost of a vacancy increases in the job type, it becomes more costly to create vacancies of higher jobs for a worker to switch to. Job upgrading is not front-loaded in general. Because job switching destroys the investment made in upgrading jobs, it is socially efficient to delay job upgrading until the job switching rate becomes small. This delay can be significant if the job switching rate is sufficiently high initially and falls quickly as the job type increases. In this case, job upgrading is hump shaped over the job type. At low job types, job upgrading is increasing because the job switching rate is declining. At high job types, job upgrading is falling because the marginal benefit of upgrading diminishes.

Job upgrading and on-the-job search induce dispersion in output among identical workers. They allow productivity to grow in a worker's career. They also increase output dispersion by reducing the option value of not taking up a job, thereby inducing the starting job type to fall. Despite the existence of job upgrading, all output dispersion in this model is caused by search frictions. All workers are identical, and the job type is not portable. Moreover, if there were no cost to maintain a vacancy or if a worker could be matched instantaneously, the social planner would create only the final job type at which output is maximized, which would eliminate output dispersion.

When the model is calibrated, the vacancy cost function needs to be highly convex in the job type in order to match the high job finding rate of unemployed workers in the data. The job-to-job transition rate is relatively small and on-the-job search stops after one switch. The job type increases in a worker's career mostly because of job upgrading. Also, since the job-to-job transition rate is small, job upgrading is front-loaded.

The calibrated model generates the ratio of the mean to the minimum in output as 2.04. This mean-min ratio can be interpreted as frictional wage dispersion by assuming that wages are proportional to output. The dispersion is close to the value in the U.S. data and much larger than in Hornstein et al. (2011). It is remarkable that the calibration meets the same targets as used by Hornstein et al. (2011) on an unemployed worker's job finding rate and home production. Since these two targets are at high values, most search models imply that the option value of continuing to search is low which, in turn, requires wage dispersion to be small. For these models to generate frictional dispersion close to the observed one, either the unemployment rate should be several times as high as the value in the data or the value of home production should be negative. The current model does not need such unrealistic parameter values to generate large output dispersion. This success shows that search frictions can be quantitatively important for generating output dispersion if they interact with job upgrading and on-the-job search.

I conduct counterfactual experiments on the convexity of the vacancy cost in the job type, job upgrading, and on-the-job search. After eliminating one ingredient at a time, I recalibrate the model. When the vacancy cost is linear in the job type, almost all dispersion in output vanishes. When job upgrading is not available, the mean-min ratio falls to 1.46. When on-the-job search is eliminated, the mean-min ratio falls only slightly to 1.996. These experiments demonstrate that the convexity of the vacancy cost in the job type is critical: Without it, the efficient allocation calls for little output dispersion even though jobs can be upgraded and workers can search on the job. With the convex vacancy cost, either job upgrading or on-the-job search can induce significant output dispersion, although job upgrading is more potent than on-the-job search. In particular, the convex vacancy cost enables on-the-job search to induce wider dispersion in this model than in Burdett and Mortensen (1998) who model on-the-job search with a single job type. In the latter model, the mean-min ratio in wages is between 1.16 and 1.27 (see Hornstein et al., 2011), which is smaller than 1.46 that on-the-job search generates in the current model.

The convex vacancy cost is important for gradual dynamics in a worker's job type and output. Without the convexity, it would be socially efficient to make the starting job type so high that would leave little room for job upgrading or on-the-job search. The convex vacancy cost can be justified by the implied dynamics in a worker's job type and output. Using personnel data, Baker et al. (1994) and Lazear and Oyer (2004) find that a typical worker spends a long time in a firm to climb up the job ladder. Part of this slow climb is caused by learning, which is abstracted from in this paper.[3] Another important cause is simply that higher level jobs are scarce and they need to be made available for workers at a lower level to ascend. Specifically, the job structure in the firm studied by Baker et al. (1994) was remarkably stable in 20 years even though the firm tripled in size. The convexity of the vacancy cost in the job type seems a convenient way to capture this climbing on the job ladder. In a related paper, Kaas and Kircher (2015) assume that the vacancy cost is convex in firm size and show that such convexity is necessary for generating dynamics in firm size. I study the dynamics of the job type instead of firm size and assume that the vacancy cost is convex in the job type rather than firm size.

Because the investment in job upgrading is specific to a match, the theory in this paper is related to the literature on match-specific training, e.g., Wasmer (2006) and Lentz and Roys (2015). In this literature, as well as the one on general training (e.g., Acemoglu and Pischke, 1999), the focus is on whether the market can internalize the pecuniary externality of training in the presence of search frictions. The current paper focuses on the constrained social optimum instead of an equilibrium. This focus increases the challenge to address the issues. For example, one cannot use market inefficiency to answer the question why job switching may be socially efficient even when job upgrading is available. Moreover, the job type in this model differs from match-specific training in an important aspect: While training can happen only after a match is formed, the job type can be created for a vacancy before a match forms. Whether it is socially efficient to create a high job type

[3] Jovanovic (1979) is a classic on the interaction between search and learning about a worker's ability.

at the beginning or upgrade a job to a high type later depends on search frictions and especially the vacancy cost. As said above, this feature of the job type enables me to attribute all output dispersion in the model to search frictions.

This paper is also related to the literature on search on the job, which has examined both undirected search (e.g., Burdett and Mortensen, 1998) and directed search (e.g., Shi, 2009, Menzio and Shi, 2011). Some of them have incorporated wage-tenure contracts, e.g., Burdett and Coles (2003) and Shi (2009). Since this literature has not allowed job upgrading, it is not able to examine the dynamic interaction between job switching and job upgrading. Moreover, in this literature, on-the-job search generates small dispersion in wages (see Hornstein et al., 2011). There are notable extensions. Tsuyuhara (2014) incorporates moral hazard into an on-the-job search model with wage-tenure contracts. His model generates a mean-min ratio of wages of 1.4, which is close to the value generated here by on-the-job search when job upgrading is not available. Lise and Robin (2014) introduce ex ante and ex post heterogeneity on both sides of a market to study dynamic sorting, which can also increase dispersion.

A large part of this paper characterizes the efficient allocation formally. This is necessary because even the basic properties, such as existence and continuity of the social value function, cannot be presumed in the presence of job upgrading and on-the-job search. In addition, the value function is not necessarily concave, which adds complexity to the analysis of the policy functions. By overcoming these difficulties, the formal characterization adds value beyond the particular subject of this paper.

2. The Model

2.1. Model Environment

Time is continuous. All workers are identical. Jobs are heterogeneous in the type denoted $h \geq 0$. A worker employed at a type h job produces a flow of output $f(h)$, where $f(0) = 0$ and $f''(h) < 0$ for all h. There exists a *final job type* $h^* < \infty$ such that $f'(h) > 0$ for all

$h < h^*$ and $f'(h) < 0$ for all $h > h^*$. The job type h^* maximizes output.[4] The job type is specific to a match. If a worker leaves a job, neither the worker nor the firm retains any part of the job type. For an unemployed worker, home production is expressed as an equivalent level of market production, $f(h_u)$, and so h_u is an unemployed worker's equivalent "job type". Assume $h_u \in [0, h^*)$.

A firm can upgrade a job by incurring the flow cost $c(i)$, where i is the upgrading rate. Let t denote a worker's tenure in the current firm. Then,

$$\dot{h}(t) \equiv \frac{\mathrm{d}h(t)}{\mathrm{d}t} = i. \tag{2.1}$$

The upgrading cost satisfies $c(\infty) = \infty$, $c(i) = c'(i) = 0$ for all $i \leq 0$, $c'(i) > 0$ for all $i > 0$, and $c''(i) > 0$ for all $i \geq 0$. The assumption $c(i) = c'(i) = 0$ for all $i \leq 0$ reflects free disposal of a job type, while other assumptions are common for adjustment costs.[5] There is no upgrading or depreciation of an unemployed worker's equivalent job type, and so an unemployed worker's home production stays constant over time.

A firm can create a vacancy of any job type. The flow cost of a type h vacancy is $\psi(h)$, where $\psi'(h) > 0$ and $\psi''(h) \geq 0$ for all $h > 0$, and $\psi(0) = 0$. As emphasized in the introduction, the ability to create vacancies of any job type distinguishes the job type from match-specific training that can only take place after a match is formed. Search frictions are the elements that may make it socially inefficient to create only the highest job type, thereby rendering job upgrading necessary.

An employed worker can search on the job with probability λ and an unemployed worker can search with probability one. Search is directed into submarkets that can be interpreted as locations. A submarket is described by (ϕ, p), where ϕ is the job type and p is the matching rate for a worker. The tightness in a submarket is denoted θ, which is the ratio of vacancies to workers searching in the submarket. The matching rate for a worker in a submarket with tightness θ is $P(\theta)$. It is convenient to invert $P(\theta)$ as $\theta(p)$ and use

[4] One justification for such hump shaped output is that production costs in addition to the labor cost increase in the job type.

[5] An example of c is $c(i) = (i + a)^\gamma - a^\gamma - \gamma a^{\gamma-1} i$ if $i \geq 0$ and $c(i) = 0$ if $i < 0$, where $\gamma > 1$ and $a > 0$. When $\gamma = 2$, a can be 0.

p as part of the description of a submarket. In a submarket with a matching rate p for a worker, the matching rate for a vacancy is $q(p) = \frac{p}{\theta(p)}$. The function $\theta(p)$ is assumed to satisfy: $\theta(0) = 0$, $\theta'(0) \in (0, \infty)$, $\theta'(p) > \frac{\theta(p)}{p} > 0$ and $\theta''(p) > 0$ for all $p > 0$, and $\lim_{p\to\infty} \frac{\theta(p)}{p} = \infty$.[6] Two examples of the matching function are given below:

Example 2.1. *The urn-ball matching function is* $P(\theta) = p_0\theta\left(1 - e^{-1/\theta}\right)$ *and the generalized telephone matching function is* $P(\theta) = \left[(\theta/p_0)^{-\rho} + 1\right]^{-1/\rho}$, *where* $p_0, \rho \in (0, \infty)$ *are constants. Both functions satisfy the assumptions on* $\theta(p)$ *above.*

An employed worker separates from a job exogenously at the rate $\delta \in (0, \infty)$, in addition to endogenous separation caused by job switches. Workers and firms have the same time discounting rate $r \in (0, \infty)$.

A clarification on the relationship between a job and a firm may be useful. As most models in the search literature, this paper treats different jobs in the same firm independently. Although a job switch is interpreted as a switch between firms, it may also be a switch within a firm if firm size is explicitly modeled. Job upgrading differs from external hiring in that no vacancy cost is incurred. As an empirical matter, a job opening at any level in the firm can be filled by either an insider or an outsider (see Baker et al., 1994, and Lazear and Oyer, 2004).

2.2. Planner's Problem

The planner is constrained by search frictions that the matching rates are finite and that the matching process cannot depend on a worker's identity. In particular, the planner's allocation must be the same for all workers employed in the same job type.[7] For a worker employed at a job type h, the planner chooses the job upgrading rate, $i(h)$, and the policy functions for job search, $(\phi(h), p(h))$. The function $\phi(h)$ is the target job type to be searched for and $p(h)$ the matching rate. Let $V(h)$ be the social value function of a match

[6]The assumptions on $\theta(p)$ are equivalent to the following assumptions on $P(\theta)$: $P(0) = 0$, $P'(0) \in (0, \infty)$, $0 < P'(\theta) < \frac{P(\theta)}{\theta}$ and $P''(\theta) < 0$ for all $\theta > 0$, and $\lim_{\theta\to\infty} \frac{P(\theta)}{\theta} = 0$.

[7]If the planner could target each worker to a specific vacancy, the planner would eliminate the matching frictions altogether by creating a separate submarket for each individual.

with the job type h and V_u the social value of an unemployed worker. I focus on the steady state where $V(h)$ depends on time only through h. That is, if $i = 0$ over time, so is $V(h)$. Then,

$$(r + \delta) V(h) = f(h) + \delta V_u + \lambda s(h) + \upsilon(h), \tag{2.2}$$

where $s(h)$ is the *social return on job search* and $\upsilon(h)$ the *social return on job upgrading.* Efficient job search solves the following problem:

$$s(h) \equiv \max_{(\phi,p)} \{p[V(\phi) - V(h)] - \psi(\phi)\theta(p)\}. \tag{2.3}$$

Efficient upgrading of the job type solves the following problem:

$$\upsilon(h) \equiv \max_i \left[\frac{\mathrm{d}V(h)}{\mathrm{d}t} - c(i)\right] \text{ s.t. (2.1).} \tag{2.4}$$

If $V'(h)$ exists, this problem of job upgrading becomes

$$\upsilon(h) \equiv \max_i [V'(h)i - c(i)]. \tag{2.5}$$

For an unemployed worker, the value of unemployment V_u is constant over time because home production is so. Precisely, V_u satisfies (2.2) with $h = h_u$ and $\dot{h}_u = 0$:

$$rV_u = f(h_u) + s_u. \tag{2.6}$$

The number s_u is defined by (2.3) with $[V(\phi) - V_u]$ replacing $[V(\phi) - V(h)]$. Denote efficient job search for an unemployed worker as (ϕ_u, p_u).

3. Efficient Allocation

Because output is decreasing in the job type for all job types higher than h^*, it is never socially efficient to have $h > h^*$. If the initial job type happens to be above h^*, the social planner will dispose part of the job type to reduce h to h^*. Without loss of generality, I will assume $h \in [0, h^*]$ throughout the analysis.

3.1. Efficient Job Switching and Job Upgrading

The functional equation (2.2) is the Bellman equation of the social value function V. To characterize the efficient allocation, it is necessary to show that the value function exists. Appendix A accomplishes this task and establishes the following lemma:

Lemma 3.1. *There exist a unique function $V(h)$ and a unique number V_u that satisfy (2.2) and (2.6). The derivative $V'(h)$ exists and is given by*

$$V'(h) = c'(i) \text{ for both } i > 0 \text{ and } i = 0, \tag{3.1}$$

where $i = i(h)$. Thus, $V(h)$ is increasing, and strictly so if $i(h) > 0$. Moreover, whenever $\frac{di}{i\ dt}$ exists, the derivative $V''(h)$ exists and is given by

$$V''(h) = c''(i) \frac{di(h(t))}{i\ dt}. \tag{3.2}$$

Thus, $V(h)$ is strictly concave near h if and only if $\frac{di}{i\ dt} < 0$.

The social value of a match is an increasing function of the job type, partly because of the assumption of free disposal. If the social value function were strictly decreasing in an interval of job types below h^*, the planner could eliminate this decreasing segment by disposing part of the job type. This modification would increase the social value and produce an increasing value function. Similarly, if job upgrading is strictly positive, then the value function must be strictly increasing in the job type.

Condition (3.1) is the first-order condition of i, which equates the marginal benefit and cost of job upgrading. The value function is differentiable because job upgrading can take place in continuous time and the marginal cost of upgrading is continuous. Any small increase in the job type can be achieved by a finite i in a small interval of time. If the value function were not differentiable at any particular job type, small upgrading near this particular job type would yield a discrete change in the social value of a match, which would contradict continuity of the social value function.[8] Note that (3.1) holds as equality

[8]This argument for differentiability of the value function uses only continuity of the value function, instead of an envelope condition as in a typical analysis (e.g., Stokey et al., 1989).

even at the corner $i = 0$, as well as for any interior $i > 0$, because $V'(h) \geq 0 = c'(0)$ while the complementary slackness condition for $i = 0$ requires $V'(h) \leq c'(0) = 0$.

Concavity of the social value function and monotonicity of job upgrading are tightly related by (3.2). The social value function is strictly concave if and only if job upgrading is decreasing over tenure. This result is intuitive. If job upgrading has diminishing marginal gains in the social value, it is socially efficient for job upgrading to taper off over tenure. If job upgrading has increasing marginal gains in the social value, it is socially efficient for job upgrading to increase over tenure to explore such increasingly larger marginal gains.

To characterize the efficient allocation in more detail, I focus on the natural case where $\frac{\mathrm{d}i}{\mathrm{d}t}$ exists. The following proposition is proven in Appendix B:

Proposition 3.2. *Assume that* $\frac{\mathrm{d}i}{\mathrm{d}t}$ *exists. The following results hold:*
(i) The derivatives $V''(h)\, i$ *and* $v'(h)$ *exist, with* $v'(h) = V''(h)\, i(h)$.
(ii) The derivative $s'(h)$ *exists and is given as* $s'(h) = -p(h)\, V'(h)$.
(iii) The envelope condition on (2.2) holds as

$$[r + \delta + \lambda p(h)]\, V'(h) = f'(h) + V''(h)\, i(h)\,. \tag{3.3}$$

Efficient upgrading changes with tenure according to

$$\frac{di}{dt} = \frac{1}{c''(i)} \left\{ [r + \delta + \lambda p(h)]\, c'(i) - f'(h) \right\}. \tag{3.4}$$

Result (i) is the envelope condition of the efficient upgrading problem. If the social value function is concave, the marginal return on job upgrading diminishes as the job type increases. Result (ii) is the envelope condition of the efficient job search problem, which shows that the marginal return on job search diminishes as the job type increases.[9]

[9] The proof of result (i) uses the fact that efficient upgrading at each h is uniquely given by (3.1) and, hence, the job upgrading policy function $i(h)$ is continuous. In contrast, the proof of (ii) does not rely on continuity of the search policy functions $(\phi(h), p(h))$, because these functions are yet to be shown to be unique for each h. Instead, the proof of (ii) first uses differentiability of $V(h)$ to establish the existence of $s'(h)$ and then uses $s'(h)$ to prove the envelope condition. With (i), (ii) and the existence of $V'(h)$, every term in (2.2) is differentiable, and (3.3) is the envelope condition on (2.2). The dynamic equation for i, (3.4), arises from substituting $V'(h) = c'(i)$ and $V''(h)\, i = c''(i)\, \frac{\mathrm{d}i}{\mathrm{d}t}$ in (3.3).

Since $c'(i) = V'(h)$, (3.4) reveals that job upgrading increases with tenure if and only if $[r + \delta + \lambda p(h)] V'(h) > f'(h)$. To explain this result, it is useful to treat a marginally higher job type as an asset. The amount, $[r + \delta + \lambda p(h)] V'(h)$, is the "permanent income" of this asset, discounted by the job separation rate $[\delta + \lambda p(h)]$ and time discounting. The cash flow of this asset is $f'(h)$, i.e., the value added by a marginally higher job type. If the permanent income of this asset exceeds the cash flow of the asset, the difference must be caused by a capital gain in the asset. In turn, for a job type to have a capital gain, the amount of upgrading must be increasing over tenure. Similarly, if the permanent income of a higher job type is less than the cash flow, there is a capital loss in the asset, in which case job upgrading declines over tenure.

Whether job upgrading increases or decrease over tenure depends on job search. The endogenous job-to-job transition rate appears in (3.4) as part of the effective discounting rate on the current job. To characterize efficient job search, let (i^*, h^*, p^*) denote the *final state* of the efficient allocation, which satisfies $\frac{\mathrm{d}i}{\mathrm{d}t} = 0 = \frac{\mathrm{d}h}{\mathrm{d}t}$. Then,

$$i^* = p^* = 0, \text{ and } f'(h^*) = 0.$$

Note that h^* is consistent with the earlier definition. The following lemma characterizes the policy functions of efficient job search (see Appendix C for a proof):

Lemma 3.3. *Consider any $h \in (0, h^*)$. (i) If $p(h) > 0$, then $\phi(h)$ satisfies $\phi(h) \in (h, h^*)$ and the first-order condition*

$$V'(\phi(h)) = \psi'(\phi(h)) \frac{\theta(p(h))}{p(h)}. \tag{3.5}$$

Also, ϕ_u satisfies this equation given $p = p_u$.

(ii) Given the efficient choice ϕ, the efficient choice $p(h)$ satisfies:

$$V(\phi(h)) - V(h) - \psi(\phi(h)) \theta'(p(h)) \leq 0 \text{ and } p(h) \geq 0, \tag{3.6}$$

where the two inequalities hold with complementary slackness. Similarly, given ϕ_u, p_u satisfies this condition with $V(h) = V_u$.

(iii) If $p(h) > 0$, then $V''(\phi) < \psi''(\phi) \frac{\theta(p(h))}{p(h)}$ at $\phi = \phi(h)$. If ψ is linear, then $V(\phi)$ is strictly concave at $\phi = \phi(h)$ whenever $p(h) > 0$.

If the efficient job switching rate is strictly positive for a worker at h, then the efficient search target must be interior. The reason is that the marginal benefit of a higher job type is zero at the final job type h^*. The marginal cost of search would exceed the marginal benefit if the search target were set as h^*, which would not be socially efficient. When the efficient search target is interior, it satisfies the first-order condition, because the value function is differentiable. In addition, the net marginal benefit of increasing the job switching rate must be non-positive, and it must be zero if the job switching rate is strictly positive. This complementary slackness condition is (3.6). If $p(h) > 0$, the return on search must be strictly concave in the search target ϕ; otherwise, a small reduction in ϕ accompanied by an increase in p can increase the return on search. This result implies that if the vacancy cost is linear in the job type, then the social value function is strictly concave at the search target whenever $p(h) > 0$.

To characterize efficient job search and upgrading, define h_c and h_T as follows:

$$V'(h_c) = \psi'(h_c)\,\theta'(0) \tag{3.7}$$

$$V(h_T) = V(h_c) - \psi(h_c)\,\theta'(0)\,. \tag{3.8}$$

Recall that $\theta'(0) > 0$. The following proposition holds (see Appendix D for a proof):

Proposition 3.4. *Assuming that $p(h)$ exists and is unique for every h, then (i) and (ii) below hold:*[10]

(i) For all $h \in (0, h^*)$ such that $p(h) > 0$, $\phi(h) \in (h, h^*)$ is unique, $\phi'(h) > 0$, and $p'(h) < 0$.

(ii) $p(h) > 0$ if and only if $h < h_T$. Moreover, $h_T < h_c = \phi(h_T) < h^*$.

In addition, assuming that $\frac{di}{dt}$ exists, then (iii) and (iv) below hold:

(iii) $i(h(t)) > 0$, $\frac{di(h(t))}{dt} < 0$, and $V(h)$ is strictly concave for all $h(t) \in [h_T, h^*)$.

(iv) If ψ is linear, the features in (iii) also hold for all $h(t) \in [\phi_u, h_T]$. If ψ is sufficiently convex, $\frac{di(h(t))}{dt} > 0$ may occur for some $h(t) \in [\phi_u, h_T]$.

[10] Appendix D also gives a sufficient condition for $p(h)$ to be unique for every h in the case where V is concave or linear.

The efficient search target is determined by (3.5), given the efficient job switching rate. The search target is an increasing function of a worker's current job type. That is, the higher a worker's current job type, the higher the target of search for the next job. The efficient job switching rate decreases in a worker's current job type. This is also intuitive. If a worker already has a high job type, further gains from increasing the job type are small. For such job switching, it is socially efficient to create only a small number of vacancies, which result in a low job switching rate.

Job switching stops after a finite number of switches. The highest job type at which a worker switches jobs is arbitrarily close to and below h_T. Such a worker searches for the job type h_c, which is strictly below h^*. At and above h_T, there is no job switching and, instead, the job is upgraded continuously toward h^*. On-the-job search stops after a finite number of job switches because of the assumption $\theta'(0) > 0$. Equivalent to $P'(0) < \infty$, this assumption requires that the marginal increase in a worker's matching rate should be bounded even when the number of vacancies in a submarket increases marginally from the initial level 0. Under this assumption, the gain from a job switch must be bounded below by a strictly positive number in order to justify the creation of vacancies for the switch, no matter how small the number of vacancies is. This lower bound on $[V(\phi) - V(h)]$ implies that job switches must stop in a finite number of steps at a job type strictly below the final level. If $\theta'(0) = 0$, contrary to the assumption, it may be possible that both the gain from and the cost of a job switch approach 0 simultaneously as the job type increases toward h^*, in which case job switches can continue indefinitely.[11]

The convergence of the job type to h^* is asymptotic. That is, job upgrading is strictly positive and declines over tenure for all finite t. To see why these features arise, suppose that job upgrading becomes constant in some interval around tenure t_0 before the job type reaches h^*. Then $(r+\delta) c'(i) = f'(h) > 0$ at t_0. This implies $i > 0$ at t_0, and so the job type will increase at t_0. Because the marginal productivity of the job type is diminishing, then $(r+\delta) c'(i) > f'(h)$ at tenure slightly higher than t_0, say $t_0 + \varepsilon$, where $\varepsilon > 0$ is

[11]This assumption $\theta'(0) > 0$ is not vacuous: Although it is satisfied by the examples in Example 2.1, it is violated by the Cobb-Douglas matching function.

arbitrarily small. This result implies that job upgrading is increasing in tenure at $t_0 + \varepsilon$. Since the marginal cost of upgrading is strictly increasing and the marginal productivity is strictly decreasing, forward induction yields $\mathrm{d}i/\mathrm{d}t > 0$ for all tenure lengths higher than t_0. As a result, job upgrading is bounded above $i\left(h\left(t_0\right)\right)$, which is strictly positive. The job type will surpass h^* in a finite length of time and will keep increasing. Because $f'\left(h\right) < 0$ for all $h > h^*$, this path of the job type cannot be socially efficient.

If the vacancy cost is linear in the job type, then job upgrading is positive and declines over tenure also for $h \in [\phi_u, h_T]$. The reason is that the social value function is strictly concave in the job type when the vacancy cost is linear in the job type (see Lemma 3.3). As the marginal value of a higher job type diminishes, the return on upgrading diminishes, and so the efficient amount of upgrading falls.

If the vacancy cost is strictly convex, it is possible that job upgrading can increase in tenure when the job type lies in $[\phi_u, h_T)$. This possibility arises if and only if the social value function is convex and, by (3.3), if and only if

$$[r + \delta + \lambda p\left(h\right)] V'\left(h\right) > f'\left(h\right). \tag{3.9}$$

This condition summarizes the conflict between two opposite forces on efficient job upgrading. One is the gain from front-loading job upgrading. The earlier is a job upgraded, the higher the joint gain in the present value of the higher productivity. This force exists at all job types and is captured by $f'\left(h\right)$ in (3.9).

The opposing force is created by job switching. When a worker leaves one job for another, all upgrading occurred in the former match is lost. Thus, job switching increases the effective discount rate on the value of a match, as shown by the presence of $p\left(h\right)$ in (3.9). To reduce this loss, the planner can delay upgrading until the job switching rate falls sufficiently. The curvature of the vacancy cost function is an important determinant of which of the two forces dominates because it affects how quickly the job switching rate falls with tenure. The more convex is the vacancy cost function, the steeper the marginal cost of a higher type vacancy, and the more quickly the job switching rate falls with tenure. In this case, the social cost of delaying job upgrading is relatively small, and so job upgrading can increase over tenure at low job types. On the other hand, when the vacancy cost is linear, the benefit of delaying job upgrading is dominated by the benefit of upgrading the job type early on, and so job upgrading decreases in tenure at all job types greater than or equal to ϕ_u.

The dynamic pattern of the job type can be summarized as follows. Job upgrading occurs in the entire duration of a worker's employment, but job switches occur only a finite number of times and in the earlier part of employment when the job type is relatively low. When an unemployed worker becomes employed, the job type is the lowest and the job switching rate the highest among all employed workers. If such a worker stays employed, the job type is also upgraded smoothly. As the job type increases because of either job switching or upgrading, the target job type of further search rises and the job switching rate falls. Job upgrading can either increase or decrease in the early stage of employment, depending on the convexity of the vacancy cost function. After a finite number of job switches, job switching stops but job upgrading continues. The amount of job upgrading declines over tenure, and so the job type increases asymptotically toward the level h^*. If the worker ever becomes unemployed, this process starts anew.

3.2. Schedule of Job Types and Effective Tenure

The dynamic pattern of the job type can be illustrated by tracing job upgrading and job search over a worker's tenure. Consider a hypothetical baseline worker who will never succeed in moving to another job despite job search. Denote the worker's tenure in the firm as t and use the subscript b to indicate a baseline worker. Immediately after exiting unemployment into a job, the worker's job type is $h(0) = \phi_u$. The path of efficient job upgrading is $\{\hat{\imath}_b(t) : t \geq 0\}$, and the path of efficient job search is $\{(\hat{\phi}_b(t), \hat{p}_b(t)) : t \geq 0\}$, where ˆ distinguishes the functions of t from the related functions of h.[12] There exists a positive and finite T such that the job switching rate is positive if and only if tenure is less than T; that is, $\hat{p}_b(t) > 0$ if and only if $t < T$. The job switching rate declines over tenure,

[12] Although a baseline worker will never succeed in moving to another firm, it is still necessary to describe where the worker should search for the next job at every tenure t.

provided that this rate is positive. Although the hypothetical worker can have a positive rate of moving to another job, the worker never gets such a lucky draw.

The baseline schedule is valid for a baseline worker not only from the start of employment, but also from any tenure length. If the worker has stayed in the firm for a length of time t_0, then the efficient allocation will be given by the "tail" of the baseline schedule from t_0 onward, which is $\{(\hat{\imath}_b(t), \hat{\phi}_b(t), \hat{p}_b(t)) : t \geq t_0\}$. This result comes from the feature that the efficient allocation is time consistent.

The baseline schedule is also useful for describing the path of efficient upgrading and search for any arbitrary worker who succeeds in job switches. Any job type h possibly reached through a sequence of job switches and job upgrading is equivalent to tenure $\tau(h)$ on the baseline schedule, where $\tau(h)$ is defined by

$$h = h_b(0) + \int_0^{\tau(h)} \hat{\imath}_b(t)\,\mathrm{d}t.$$

Let me refer to $\tau(h)$ as the effective tenure of a worker at a job type h. Given h, if the worker will stay in the current job forever until exogenous separation, then the efficient allocation for the worker in the future is:

$$\{(\hat{\imath}(t), \hat{\phi}(t), \hat{p}(t)) : t \geq 0\} = \{(\hat{\imath}_b(t+\tau), \hat{\phi}_b(t+\tau), \hat{p}_b(t+\tau)) : t \geq 0\},$$

where $\tau = \tau(h)$. More generally, when a worker switches from h to $\phi_b(h)$ through on-the-job search, the switch is equivalent to keeping the job in the firm and increasing the effective tenure from $\tau(h)$ to $\tau(\phi_b(h))$. After the switch, the efficient allocation for the worker is the baseline schedule from tenure $\tau(\phi_b(h))$ onward, until the worker moves again. Because the efficient schedule at any arbitrary job type can be described by using the baseline schedule, I will focus on the baseline schedule and omit the subscript b.

Figure 1 depicts the baseline schedule of the job type as the solid curve and a sample path of job switches as the dashed steps. After a worker moves out of unemployment and into employment, the job type increases along the baseline curve as the job is upgraded gradually. At tenure t_1, the worker succeeds in moving to a new firm with a job type $\phi(h(t_1))$. This move is equivalent to increasing tenure from t_1 to $t_2 = \tau(\phi(h(t_1)))$ on the baseline schedule, where the function $\tau(.)$ is defined above. This move takes the worker from the job type $h(t_1)$ to $h(t_2) = \phi(h(t_1))$. After the move, the job type for the worker increases along the baseline curve until the worker succeeds again in moving to another job. The maximum effective tenure length below which a worker moves to another job at a positive rate is $T = \tau(h_T)$, where h_T is defined by (3.8). For all effective tenure lengths greater than or equal to T, the only cause of an increase in the job type is job upgrading.

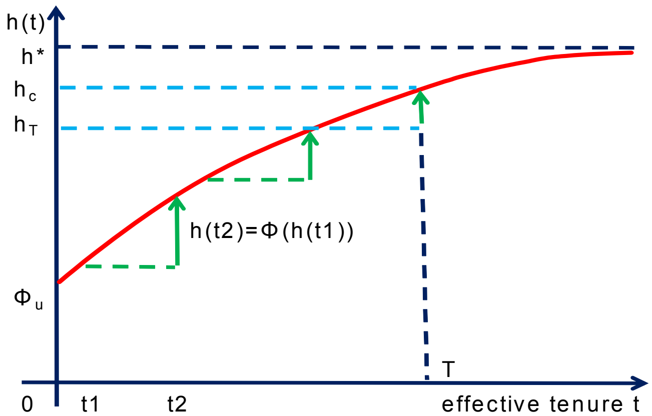

Figure 1. Efficient job upgrading and switching

4. Distribution of Workers

This section characterizes the distribution of workers in the final state. Let n_u be the measure of unemployed workers and $n_e = 1 - n_u$ be the measure of employed workers. Because an unemployed worker finds a job at the rate p_u and an employed worker separates from a job into unemployment at the rate δ, the measure of unemployed workers remains constant over time if and only if $n_u p_u = n_e \delta$. This equation solves

$$n_u = \frac{\delta}{\delta + p_u}.$$

Let t denote a worker's actual tenure. In contrast to the effective tenure, the actual tenure is reset to zero whenever a worker switches a job. Denote the cumulative distribution of employed workers over (h, t) as $\Omega(h, t)$, and the corresponding density function as $\omega(h, t)$. For any $h \geq \phi_u$ and $t \geq 0$, the measure of workers with $(h, t) \in (h, h^*] \times (t, \infty)$ is

$$\bar{\Omega}(h, t) \equiv \int_h^{h^*} \int_t^{\infty} \omega\left(\tilde{h}, \tilde{t}\right) \mathrm{d}\tilde{t}\mathrm{d}\tilde{h}. \tag{4.1}$$

This is less than $1-\Omega(h,t)$, because there are workers with $(h,h^*]$ and $[0,t]$ and workers with $[\phi_u,h]$ and (t,∞). Denote the partial derivatives of $\bar{\Omega}$ as

$$\begin{aligned}\bar{\Omega}_h(h,t) &\equiv -\textstyle\int_t^{\infty}\omega\left(h,\tilde{t}\right)\mathrm{d}\tilde{t}\\ \bar{\Omega}_t(h,t) &\equiv -\textstyle\int_h^{h^*}\omega\left(\tilde{h},t\right)\mathrm{d}\tilde{h}.\end{aligned} \tag{4.2}$$

The amount $-\bar{\Omega}_h(h,t)$ is the measure of workers employed at the job type h over all tenure greater than t, and the amount $-\bar{\Omega}_t(h,t)$ is the measure of workers employed at tenure t over all job types greater than h. Clearly, $\bar{\Omega}_{ht}(h,t)=\bar{\Omega}_{th}(h,t)=\omega(h,t)$.

Let $G(h)$ be the cumulative distribution of employed workers with job types at or lower than h summed over all lengths of actual tenure, and let $g(h)$ be the corresponding density. Similarly, let $G_t(t)$ be the cumulative distribution of employed workers with tenure less than or equal to t, summed over all job types, and let $g_t(t)$ be the corresponding density function. The distribution of employed workers is characterized by the following proposition (see Appendix E for a proof):

Proposition 4.1. *Assume that the joint density of employed workers,* $\omega(h,t)$*, exists for all* $(h,t)\in[\phi_u,h^*]\times[0,\infty)$*. Then* $\omega\left(\tilde{h},t\right)=0$ *for all* $\tilde{h}<h(t)$ *and* $\omega\left(h,\tilde{t}\right)=0$ *for all* $\tilde{t}>t(h)$*, where* $t(h)$ *is the inverse function of* $h(t)$*. For all* (h,t)*, the joint distribution of employed workers obeys:*

$$i(h)\,\bar{\Omega}_h(h,t)+\bar{\Omega}_t(h,t)=\int_h^{h^*}\left[\delta+\lambda p\left(\tilde{h}\right)\right]\bar{\Omega}_h\left(\tilde{h},t\right)d\tilde{h}. \tag{4.3}$$

For all $h\in[\phi_u,h^*]$*, the marginal distribution of employed workers over* h *obeys:*

$$i(h)\,g(h)=\delta\left[1-G(h)\right]+\bar{\Omega}_t(h,0)+\int_h^{h^*}\lambda p\left(\tilde{h}\right)g\left(\tilde{h}\right)d\tilde{h}. \tag{4.4}$$

For all $t\geq 0$*, the marginal distribution of employed workers over* t *obeys:*

$$g_t(t)=\delta\left[1-G_t(t)\right]-\int_{\phi_u}^{h^*}\lambda p\left(\tilde{h}\right)\bar{\Omega}_h\left(\tilde{h},t\right)d\tilde{h}. \tag{4.5}$$

The marginal densities have the following properties: (i) $g_t(t)$ *is strictly decreasing and differentiable for all* $t\in[0,\infty)$*; (ii)* $g(h)$ *is differentiable for all* $h\in[\phi_u,h^*)$*; (iii) For all* $h\in(h_c,h^*)$*,* $\frac{d}{dh}\left[i(h)\,g(h)\right]=-\delta g(h)<0$*, and* $g(h)$ *is decreasing if and only if* $f'(h)<\delta i c''(i)+(r+\delta)\,c'(i)$ *where* $i=i(h)$*.*

If a worker never switches jobs, the job type reached at tenure t by job upgrading is $h(t)$. Because the worker can also increase the job type by searching on the job, no worker at tenure t has a job type lower than $h(t)$. That is, $\omega\left(\tilde{h},t\right)=0$ for all $\tilde{h}<h(t)$. Similarly, $\omega\left(h,\tilde{t}\right)=0$ for all $\tilde{t}>t(h)$, where $t(h)$ is the inverse function of $h(t)$.

Equation (4.3) is the steady state equation of the measure of workers with $(h,t)\in(h,h^*]\times(t,\infty)$. The amount $-i(h)\,\bar{\Omega}_h(h,t)\,\Delta$ is the inflow of workers employed slightly below h whose jobs are upgraded above h in a small time interval Δ. The amount $-\bar{\Omega}_t(h,t)\,\Delta$ is the inflow of workers with job types above h whose tenure increases above t in a small time interval Δ. The outflow is $-\Delta$ times the right-hand side of (4.3), which is generated by exogenous and endogenous separation from the jobs.

The marginal density of employed workers over job types obeys (4.4). To explain this equation, it is useful to rewrite it to involve only the marginal distribution of h. Note that the measure of workers with tenure 0 and job types above h is $-\bar{\Omega}_t(h,0)>0$. These workers must be the ones who just moved to job types above h, since their tenure is 0. Of these movers, a subgroup of workers were employed at or above h before the job switch, and their measure is given by the integral on the right-hand side of (4.4). Thus, the sum of the last two terms in (4.4) is equal to the negative of the measure of the workers who just moved to job types at or above h from jobs below h. Calculating this measure, I can rewrite (4.4) solely in terms of the distribution of h as

$$i(h)\,g(h)=\delta\left[1-G(h)\right]-\int_{\max\{\phi^{-1}(h),\phi_u\}}^{h}\lambda p\left(\tilde{h}\right)\mathrm{d}G\left(\tilde{h}\right). \tag{4.6}$$

The lower bound on the integral uses the fact that no worker is employed below ϕ_u.

Properties (i)-(iii) in Proposition 4.1 are intuitive. Because job upgrading is continuous, employed workers at any state (h,t) exit the state smoothly in an arbitrarily small interval of time. Thus, the distribution of employed workers has no mass point and the marginal densities are differentiable for all $t<\infty$ and $h<h^*$, as described by (i) and (ii). The only possible exception is the final state $(h,t)=(h^*,\infty)$, because job upgrading stops at $h=h^*$. To explain (iii), note that for all $h\in(h_c,h^*)$, job upgrading is the only process by which an employed worker can reach a higher job type. However, a worker at such a

job type can exit the job type by either job upgrading or exogenous separation. For the density of workers at such a job type to be stationary, the rate of the outflow through job upgrading must decline over h in order to balance the exogenous separation. That is, $\frac{\mathrm{d}}{\mathrm{d}h}[i(h)\,g(h)] = -\delta g(h) < 0$. Since $i(h)$ is also decreasing at such job types, $g(h)$ is decreasing if and only if $-i'(h) < \delta$, which is equivalent to $f'(h) < \delta i c''(i) + (r+\delta)\,c'(i)$ where $i = i(h)$. Therefore, the distribution of employed workers can be decreasing at high job types when the marginal productivity diminishes sufficiently quickly.

5. Quantitative Analysis

5.1. Calibration

To calibrate the model, I use the following functional forms of output, $f(h)$, the vacancy cost, $\psi(h)$, the upgrading cost, $c(i)$, and the tightness, $\theta(p)$:

$$
\begin{aligned}
&f(h) = h^{\alpha} - f_0 h, \quad f_0 > 0,\ \alpha \in (0,1);\\
&\psi(h) = \psi_0 h^{\psi_1}, \quad \psi_0 > 0,\ \psi_1 \geq 1;\\
&c(i) = c_1 i^2, \quad c_1 > 0;\\
&\theta(p) = p_0\left[p^{-\rho} - 1\right]^{-1/\rho}, \quad p_0, \rho > 0.
\end{aligned}
$$

The convexity of the vacancy cost is governed by ψ_1 and will be calibrated. Since the convexity of the upgrading cost is important only relatively to the convexity of the vacancy cost, I set the upgrading cost as the quadratic function. The function $\theta(p)$ is generalized telephone matching function in Example 2.1. I calibrate the model monthly. Table 1 lists the parameters, their values and the calibration targets.

Several aspects of the calibration are worth noting. First, an unemployed worker's monthly job finding rate is set at 0.374 and an unemployed worker's home production at 36% of the *highest* level of market production. The job finding rate comes from targeting the unemployment rate at 0.065, and home production amounts to 40% of the average market production. These two targets are the same as in Hornstein et al. (2011). The value of home production is also close to the one used by Shimer (2005) in the study of the labor market in the business cycle, but lower than the one used by Hagedorn and Manovskii (2008). As shown by Hornstein et al. (2011), setting home production as 0.4

of the average market production already puts a strong discipline on how much frictional wage inequality that a search model can generate. If home production is set to a much higher value, say 0.8, the model is a non-starter to generate frictional dispersion.

Second, the value $\rho = 0.5$ implies that the elasticity of the job finding rate with respect to the tightness of the submarket for unemployed workers is $\varepsilon_u \equiv \frac{\mathrm{d}\ln p_u}{\mathrm{d}\ln\theta(p_u)} = 0.39$. There is no empirical estimate of this elasticity. Instead, the estimate is on the elasticity of an unemployed worker's job finding rate with respect to the tightness $\bar{\theta} = \bar{v}/u$, where $\bar{v}$ is the total number of vacancies. This estimate is around 0.27 (see Shimer, 2005), although the value 0.5 is also used sometimes (e.g., Petrongolo and Pissarides, 2001). Menzio and Shi (2011) explain that the empirical estimate is lower than ε_u because there is on-the-job search and search is directed into submarkets. Their calibration yields $\varepsilon_u = 0.60$. The choice of ρ, together with the target on the job finding rate $p_u = 0.374$, yields an elasticity between these two numbers in the literature.

Table 1. Identification of the parameters

Parameter	Value	Target
r	4.15×10^{-3}	quarterly interest rate = 0.0125
α	0.64	labor share of output = 0.64
f_0	0.1219	normalize $h^* = 100$
h_u	6.2522	$f(h_u)/f(h^*) = 0.36$
δ	0.026	monthly EU rate in CPS
λ	1	benchmark
ρ	0.5	elasticity $\frac{\mathrm{d}\ln p_u}{\mathrm{d}\ln\theta_u} = 0.39$
ψ_0	4.9382×10^{-7}	$\psi(\phi_u)/f(\phi_u) = 0.10$
ψ_1	5.8922	(ψ_1, p_0, c_1) minimize the distance of
p_0	0.6246	$(p_u, \phi_u/h^*)$ from the values given by
c_1	0.0529	$p_u = 0.374$ and $f(\phi_u)/f(h^*) = 0.45$

Third, the lowest output of an employed worker is set to be 45% of the highest output, and the vacancy cost of the lowest employed job type ϕ_u is 10% of output at ϕ_u. The value of the lowest market output is reasonable and the vacancy cost is comparable to the ones in Silva and Toledo (2009).[13] Note that these two targets yield $\phi_u/h^* = 0.0963$ and

[13]They estimate that it takes 17.2 days on average to fill a vacancy and, during this time, the average time spent on recruiting by hiring supervisors is 13.5 hours. This resource spent in hiring amounts to 13% of the monthly wage of a newly hired worker. I approximate this vacancy cost as a fraction of a new worker's output instead of wages.

$\psi_0 (\phi_u)^{\psi_1} = 0.309$.

The three parameters, (ψ_1, p_0, c_1), are chosen to minimize the distance between the model and the two targets, $p_u = 0.374$ and $\phi_u/h^* = 0.0963$. The values of (ψ_1, p_0, c_1) reported in Table 1 yield $p_u = 0.379$ and $\phi_u/h^* = 0.0959$, which are close to the targets. Appendix F provides further details of the calibration, the computation of the value and policy functions, and the simulation of the steady state distribution.

5.2. Worker Mobility and Job Upgrading

In the computed model, job switching stops at $h_T = 12.164$ and the highest job type that can be reached by job switching is $h_c = 13.962$. Although these job types seem low relative to the highest job type $h^* = 100$, the output levels at these job types are significant. As fractions of the highest output, the output is 45% at the lowest job type, 50.5% at the job type h_T, and 54% at the job type h_c. Thus, on-the-job search can yield 20% of gain in output. Figure 2 depicts the policy functions of the efficient allocation.[14] In order to exhibit the search sequence clearly, the upper panel plots $f(\phi(h))$ against $f(h)$, instead of $\phi(h)$ against h. The solid line in the upper panel in Figure 2 is output at the search target, $f(\phi(h))$, as a function of a worker's current output. Confirming Proposition 3.4, this function is strictly increasing for all $h < h_T$, and it coincides with $f(h)$ for all $h > h_T$. The flat segment of the function $f(\phi(h))$ illustrates the features that a worker at h_T searches for h_c and that job search stops for $h > h_T$.[15] For all $h > h_c$, $f(\phi(h)) = f(h)$.

The dashed steps in the upper panel in Figure 2 are the sequence of output levels searched by a worker who starts in unemployment and succeeds in every job search. The first step is an unemployed worker's home production, the second step is the output level of the first job, $f(\phi_u)$, and the third step is the output level of the next job, $f(\phi(\phi_u))$. An employed worker stops searching after switching job once. That is, the job type $\phi(\phi_u)$ lies in $(h_T, h_c]$. Note that the depicted job path occurs with probability zero, because the

[14]In the computation, the search problem in (2.3) and the job upgrading problem in (2.4) are solved with grid search. The resulting policy functions $(\phi(h), p(h), i(h))$ are single-valued for each h.

[15]The flat segment can start at any $f(h) \in [f(\phi^{-1}(h_T)), f(h_T))$ and end at the corresponding target $f(\phi(h)) \in [f(h_T), f(h_c))$. These flat segments are not depicted.

probability of immediately getting another job after a job switch is zero. Almost surely, a worker will stay with a job for a positive length of time before moving to another job. During this time, the worker's job type is upgraded. If the tenure in the job is sufficiently short, the worker will search for another job and experience a job switch with positive probability. If the tenure in the job is so long that the job type is upgraded beyond h_T, the worker will stop searching. If a worker becomes unemployed, the worker will start the process of job search and job upgrading anew.

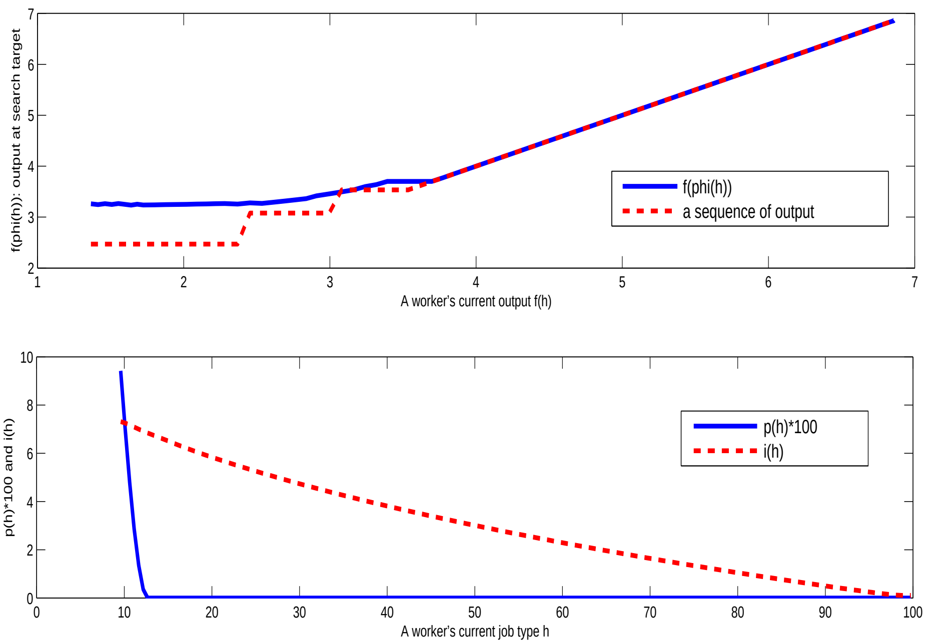

Figure 2. The policy functions $f(\phi(h)), p(h)$ and $i(h)$

The lower panel in Figure 2 depicts the policy functions of the job switching rate, $p(h)$, and the upgrading rate, $i(h)$. The job switching rate is multiplied by 100 in order to fit into the figure. This function is strictly decreasing for all $h < h_T$, reaches zero at $h_T = 12.164$ and stays at zero thereafter. The job upgrading rate is positive for all $h < h^*$ and declines over the job type. The possibility described in Proposition 3.4 that the job upgrading rate can increase at low job types does not arise with the particular parameter values. The reason is that the job switching rate is small at all job types. Since an upgraded job faces only a low probability of being destroyed by job switching, the gain from front-loading job upgrading dominates the gain from delaying job upgrading.

Figure 3 depicts the job type and the upgrading rate as functions of a worker's effective tenure. The solid line is the baseline schedule of job types. As described earlier, this is the path of the job types of a worker who never succeeds in on-the-job search. The dashed line is the job upgrading rate for such a worker, multiplied by 10. As the effective tenure increases, the job type increases and approaches the final level h^* asymptotically. After 7 years, the job type is close to this limit.

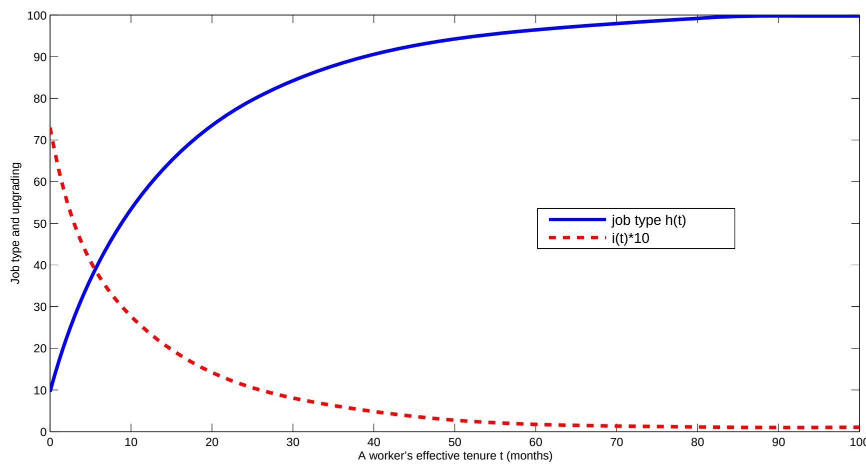

Figure 3. The job type and upgrading as functions of effective tenure

5.3. Worker Distribution and Frictional Inequality

Using the policy functions, I simulate the economy to obtain the steady state. For the simulation, time is discretized into small intervals dt and the job type is discretized according to the function $h(t)$ (see Appendix F). The top panel in Figure 4 depicts the frequency function of unemployed workers over the unemployment duration. The vertical bar over a number t is the frequency of workers whose unemployment duration lies in $[t, t+1)$ quarters. The frequency of unemployed workers declines sharply over the unemployment duration. About 80 percent of unemployed workers become employed in one quarter.

The lower panel in Figure 4 depicts the frequency function of employed workers over actual tenure (in years). A bar over a tenure length t is the frequency of workers whose actual tenure lies in $[t, t+1)$ years. The frequency of workers decreases in tenure. Twenty seven percent of employed workers have tenure less than one year, but only 20 percent of employed workers have tenure between one and two years. Exogenous separation contributes to this large drop in the frequency of employed workers from less than one year to above one year of tenure, but it is not the only cause. Another cause is that tenure is reset to zero when a worker switches jobs, which increases the flow of workers into tenure less than one year. Since there is no such inflow of workers from longer tenure into two years of tenure, there is a large drop in the measure of workers when actual tenure increases from less than one year to above one year.

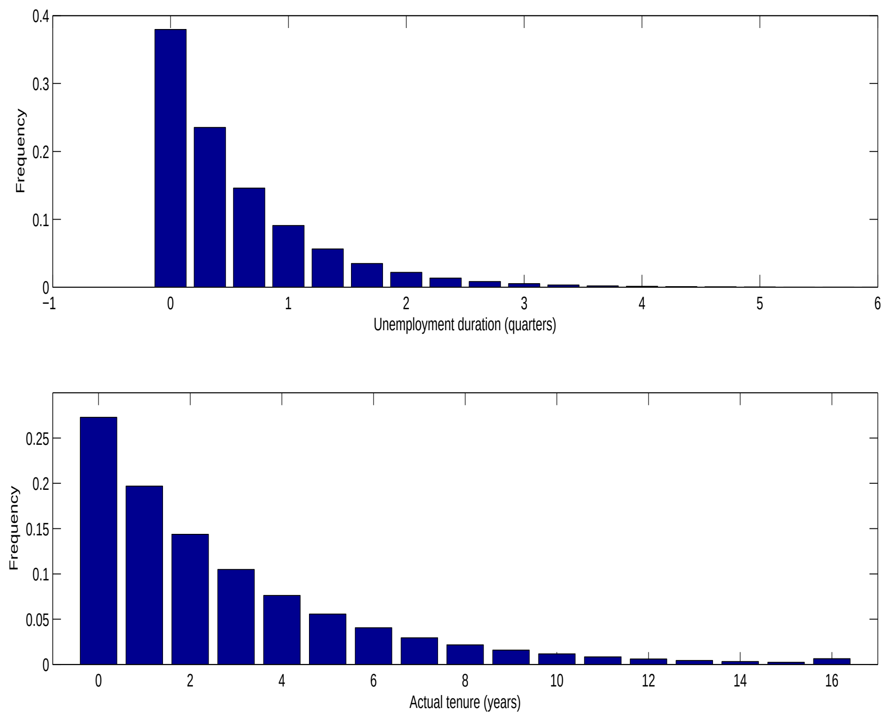

Figure 4. Frequency of unemployment duration and employed tenure

Figure 5 depicts the distribution of workers over job types (the upper panel) and output (the lower panel). Since the job type is discretized according to the function $h(t)$ given the grid of t, the frequency is positive only at a finite number of job types. Because the slope of $h(t)$ declines as t increases (see Figure 3), the gap between two adjacent job types with positive frequency becomes smaller when the job type increases. There is a spike at the final job type in the upper panel and at the final output in the lower panel, because upward mobility stops at the final job type. Excluding this spike, the frequency function of workers decreases in the job type. Relative to the frequency of job types, the frequency of output (in the lower panel) has wider gaps at low levels of output and is bunched more closely at high levels of output. These differences are caused by concavity of $f(h)$, which makes the gap between two adjacent levels of output with positive frequency shrink quickly as output increases.

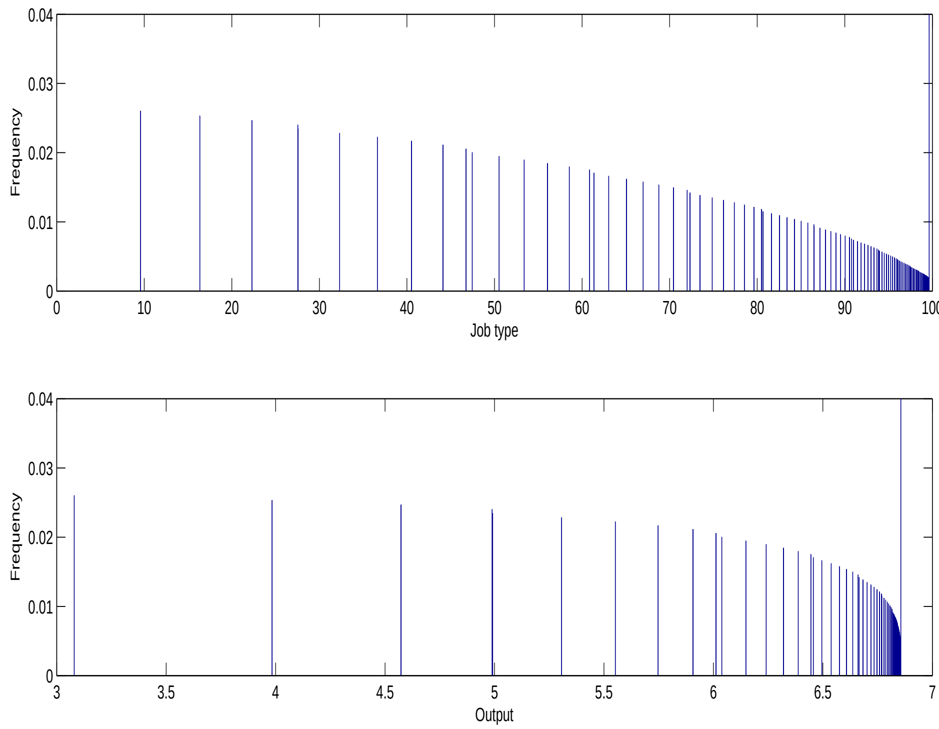

Figure 5. Frequency of employed workers' job types and output

Table 2 lists the statistics of the job type, gross output h^{α} and output $f(h)$. The dispersion in each of the three variables is large. The ratio of the mean to the minimum of the job type is 7.15. This translates into a mean-min ratio of 3.46 in h^{α} and 2.04 in output. I consider three other measures of dispersion: the coefficient of variation, the 90-10 percentile ratio, and the 50-10 percentile ratio. The coefficient of variation is 0.38 in job types, 0.27 in h^{α} and 0.14 in output. The 90-10 percentile ratio is 3.58 in the job type, 2.26 in h^{α} and 1.37 in output. The 50-10 percentile ratio is 2.76 in job types, 1.91 in h^{α} and 1.34 in output. All these numbers indicate significant dispersion, but different measures of dispersion reveal different information. The relatively small difference between the 90-50 percentile ratio and the 50-10 percentile ratio implies that the 90 percentile output is close to the 50 percentile. The mean-min ratio is significantly larger than the 50-10 percentile ratio. This difference means that there is only a small measure of workers employed at the lowest job type and that these workers quickly transit into the 10th percentile output or beyond by either on-the-job search or job upgrading.

Table 2. Statistics of job types and output

	mean	st. dev.	$\frac{\text{st. dev.}}{\text{mean}}$	min	max	90/10	50/10	$\frac{\text{mean}}{\text{min}}$
h	69.25	26.33	0.38	9.68	99.67	3.58	2.76	7.15
h^{α}	14.74	4.02	0.27	4.26	19.01	2.26	1.91	3.46
$f(h)$	6.29	0.87	0.14	3.08	6.86	1.37	1.34	2.04

To relate the dispersion in Table 2 to observed dispersion in wage rates requires an assumption on how wages are determined. A reasonable assumption is that the wage rate is a constant fraction of output. Under this assumption, the mean-min ratio, the coefficient of variation, the 50/10 percentile ratio and the 90/10 percentile ratio are the same in wages as in output. In particular, the mean-min ratio in the wage rate is 2.04. Thus, the model can capture most of wage dispersion among homogeneous workers observed in the data (see Hornstein, et al., 2011). After quantifying several versions of search models, those authors find that the models can generate no more than 1.05 as the mean-min ratio in wage rates, where the empirical estimate of the mean-min ratio is about 2.

As explained well by Hornstein et al. (2011), wage dispersion is small in most search models because of two empirical restrictions: The monthly job finding rate is relatively high and the value of an unemployed workers' home production is significantly positive. For the job finding rate to be high, an unemployed worker must be eager to take up a job rather than continue to search. That is, the option value of staying unemployed must be low, which consists of the value of home production and the future value of obtaining high offers after search. Since the value of home production is significantly positive, the option value of continuing to search for higher offers must be low in order to explain the high job finding rate. This implies that dispersion of wages is small. To generate significant wage dispersion and an empirically reasonable job finding rate, Hornstein et al. (2011) find that most search models would need the value of home production to be negative.

The calibration above satisfies both empirical restrictions in Hornstein et al. (2011). Thus, the large wage dispersion in the current model must be caused by the new ingredients in the model: the convex vacancy cost, job upgrading, and one the job search. The next subsection will investigate the quantitative role of each of these ingredients in output dispersion. Before embarking on this investigation, it is important to repeat that output dispersion in this model is entirely caused by search frictions. In particular, the role of job upgrading should be attributed to search frictions. If the vacancy cost were small or independent of the job type, job upgrading would not be necessary and, hence, would not contribute to output dispersion in the efficient allocation.

5.4. Counterfactual Experiments

The convexity of the vacancy cost, job upgrading and on-the-job search all increase output dispersion. The convexity of the vacancy cost increases the marginal cost of a vacancy and, hence, makes it socially inefficient to start with a high job type. Job upgrading and on-the-job search induce unemployed workers to accept a low job type as the starting job and provide output growth during employment. To investigate the quantitative importance of the three ingredients, I conduct counterfactual experiments by eliminating each of these ingredients in turn, recalibrating the model and computing the efficient allocation.

In the counterfactual experiment on the vacancy cost, I set $\psi_1 = 1$ so that the vacancy cost is linear in the job type. The model is recalibrated to identify (ψ_0, p_0, c_1). The parameter ψ_0 needs to be recalibrated in order to make the experiment non-trivial. If ψ_0 is kept at the baseline value, the reduction in ψ_1 from the baseline value 5.8922 to 1 will reduce the vacancy cost by so much that output dispersion will be minuscule. I set ψ_0 so that the vacancy cost at the lowest employed job type in the baseline economy is the same as under $\psi_1 = 1$. This yields $\psi_0 = 0.0314$. The parameters (p_0, c_1) are recalibrated to minimize the distance between the target on p_u and the model.[16] The target on ϕ_u/h^*

[16]The recalibrated values are $p_0 = 0.4997$ and $c_1 = 0.2485$. The job-finding rate is $p_u = 0.3695$ and the

is dropped because the lowest employed job type is an important channel through which the convexity of the vacancy cost affects output dispersion. All parameters other than $(\psi_0, \psi_1, p_0, c_1)$ are kept at their baseline values.

With linear vacancy costs, the lowest job type increases to $\phi_u = 89.74$. This is much higher than in the baseline economy because the marginal cost of creating vacancies of higher types is now lower. The ratio of the lowest output to the highest output is $f(\phi_u)/f(h^*) = 0.996$. With such high output at the lowest employed job type, there is no gain to creating vacancies of higher job types. As a result, the efficient allocation features no on-the-job search and job-to-job transitions. Similarly, job upgrading generates very little output dispersion. The mean/min ratio in output drops to 1.002. Thus, the convexity of the vacancy cost is critical for the model to generate a realistic amount of frictional dispersion in output.

In the counterfactual experiment on job upgrading, I set $c_1 = 10^6$ so that job upgrading is almost impossible. The parameters (ψ_1, p_0) are recalibrated to minimize the distance between the model and the target on p_u. All other parameters remain at their baseline values in Table 1. The target on ϕ_u/h^* is dropped again because one purpose of the experiment is to check how the the absence of job upgrading affects the lowest employed job type. The recalibrated model yields $f(\phi_u)/f(h^*) = 0.5311$.[17] The mean/min ratio in output is 1.46. The significant reduction in the mean-min ratio from the baseline economy shows that job upgrading is quantitatively important for frictional dispersion. This role can be explained by the change in the starting job type. When job upgrading is not available, on-the-job search is the only way by which a worker's job type will be increased. Since a worker starting with a low job type will be with the job for a long time, it is socially efficient to increase the starting job type. With a higher starting job type, the room for the job type to increase in the future through on-the-job search decreases. Thus, output dispersion decreases.

unemployment rate is $u = 0.0657$. The distance of p_u from the target is 0.45%.

[17]The recalibrated parameters are $\psi_1 = 5.5351$ and $p_0 = 0.4248$. The recalibrated model generates $p_u = 0.3425$, $\phi_u = 13.498$ and $u = 0.0706$. The distance in p_u from its target is $drest = 3.15\%$.

It is worth noting that the mean-min ratio in output is still significant even when job upgrading is shut down. A relevant comparison is with the on-the-job search model by Burdett and Mortensen (1998). After calibrating the latter model, Hornstein et al. (2011) find that the mean-min ratio in wages is between 1.16 and 1.27 if the job-to-job transition rate is not unrealistically high. In the above model with only on-the-job search, the monthly job-to-job transition rate is less than 2%, which is realistic. The significantly higher mean-min ratio, 1.46, can be traced to the convex vacancy cost. As explained above, a higher convexity of the vacancy cost in the job type makes it socially efficient to reduce the starting job type, thus increasing output dispersion.

Finally, I set $\lambda = 0$ to shut down on-the-job search and recalibrate (ψ_1, p_0, c_1) to minimize the distance of the model from the targets on $(p_u, \phi_u/h^*)$. All other parameters are fixed at their values in the benchmark calibration and, in particular, $\psi_0 = 4.938 \times 10^{-7}$. The recalibrated model yields $p_u = 0.373$ and $\phi_u/h^* = 0.0959$.[18] The starting job type is almost the same as in the benchmark calibration. The mean/min ratio in output is 1.996, which is only slightly lower than in the baseline economy. This result is not surprising given the finding in Figure 1. Because workers switch jobs no more than once in the baseline economy, shutting down on-the-job search does not reduce output dispersion substantially. However, this result does not mean that on-the-job search is not important for frictional dispersion in general. If job upgrading is not available, on-the-job search can induce sizable dispersion in output, as shown above in the counterfactual experiment on c_1.

6. Conclusion

A worker's job can be improved internally through job upgrading or externally through on-the-job search. Incorporating these internal and external job dynamics into a directed search model, I analytically characterize and quantitatively evaluate the socially efficient creation of vacancies, search and job upgrading. The analysis shows that efficient job upgrading continues throughout a worker's career and may be hump shaped over tenure.

[18]The recalibrated parameters are $\psi_1 = 5.5351$, $p_0 = 1.4367$ and $c_1 = 0.0529$. The unemployment rate is $u = 0.652$. The recalibrated values of $(p_u, \phi_u/h^*)$ differ from their targets by a distance $drest = 0.105\%$.

In contrast, on-the-job search is front-loaded in a worker's career and stops after a finite number of job switches. The dynamic interaction between job upgrading and on-the-job search can generate large frictional dispersion in output among identical workers. The calibrated model yields the mean-min ratio in output as 2.04, which is empirically plausible and much larger than in previous models. Both job upgrading and on-the-job search generate significant dispersion in output, although job upgrading is more potent than on-the-job search. Output dispersion depends critically on the calibrated feature that the marginal cost of a vacancy increases in the job type. If the marginal cost of a vacancy were non-increasing in the job type, the social planner would start all jobs at a high type and leave very little room for job upgrading or on-the-job search.

The formal characterization and the quantitative evaluation in this paper should both be useful for future research. To focus on frictional dispersion, I have deliberately abstracted from heterogeneity among workers and firms. It is useful to incorporate such heterogeneity and, especially, match-specific heterogeneity. In the calibrated model, the job-to-job transition is unrealistically low and on-the-job search stops after one job switch. If match-specific productivity is introduced, a match may yield productivity lower than expected, in which case the efficient allocation is to move the worker to a better match. This will increase the amount and the duration of on-the-job search and may delay job upgrading. Another useful extension is to allow the investment in job upgrading to be partly salvaged rather than completely destroyed when a worker switches firms. The higher the fraction of job upgrading that can be salvaged, the more likely that job upgrading will be front-loaded, and less frequently that a worker will switch jobs. Finally, one can study how the market may succeed or fail to internalize the externalities created by job upgrading and on-the-job search.

Appendix

A. Proof of Lemma 3.1

To start, recall that $f(h^*)$ is the maximal output. The domain of h can be set as $[0, h^*]$. The value function V is bounded above by $V^* \equiv f(h^*)/r$, and so the range of V is $[0, V^*]$. Because V is bounded, s and υ must also be bounded. This implies that the choices p and p_u must be bounded by some number $\bar{p} < \infty$.

For $\lambda \neq 1$, the functional equation of V_u differs from that of $V(h)$. Denote $\Delta V(h, V_u) \equiv V(h) - V_u$. I first fix V_u to be any arbitrary value in $[0, V^*]$ and prove that the function $\Delta V(h, V_u)$ exists, is unique, and is continuous. Then I prove that a unique number V_u exists. To derive the functional equation of ΔV, use the fact that V_u is a constant to rewrite (2.2), (2.3) and (2.4) as

$$(r+\delta)\,\Delta V(h, V_u) = f(h) - rV_u + \lambda s(h) + \upsilon(h) \tag{A.1}$$

$$s(h) = \max_{(\phi,p)} p(h)\left[\Delta V(\phi, V_u) - \Delta V(h, V_u)\right] - \psi(\phi)\,\theta(p) \tag{A.2}$$

$$\upsilon(h) = \max_i \left[\frac{\mathrm{d}\Delta V(h, V_u)}{\mathrm{d}t} - c(i)\right] \tag{A.3}$$

Since $\upsilon(h)$ involves the derivative of ΔV, so does the right-hand side of (A.1). The existence of this derivative is yet to be proven. To resolve this problem, I express the efficient allocation alternatively as a sequence problem and show that the sequence problem yields a unique value function. Because the solution to the optimization problem is time consistent, the value function generated by the sequence problem is also the unique fixed point of the Bellman equation, (2.2).

The choices in the sequence problem are functions of the effective tenure. I add the symbol $\hat{}$ to these functions to distinguish them from the functions of the job type. Let t denote the tenure of a worker in a firm. Consider a worker who has tenure $t_0 \geq 0$ in a firm and a job type $h(t_0)$. Given $h(t_0)$, the planner chooses a time path of job upgrading, $\{\hat{\imath}(t)\}_{t \geq t_0}$, and a time path of on-the-job search, $\{\hat{\phi}(t), \hat{p}(t)\}_{t \geq t_0}$. These paths are operative only when the worker stays in the firm. If the worker moves to a new match, the planner chooses new paths of job upgrading and job search. Suppose that a sequence, $\{\hat{\imath}(t), \hat{\phi}(t), \hat{p}(t)\}_{t \geq t_0}$, is efficient. Let $\{h(t)\}_{t \geq t_0}$ be the sequence of job types induced by the sequence of $\hat{\imath}$ according to (2.1), where the symbol $\hat{}$ on h is suppressed. Substitute s and υ from (A.2) and (A.3) into (A.1). For the moment, omit the two maximization operators. Denote the effective discount factor between t_0 and t as

$$D(t, t_0) = e^{-\int_{t_0}^{t}[r+\delta+\lambda\hat{p}(\tau)]\mathrm{d}\tau}. \tag{A.4}$$

For any $t_0 \geq 0$ and $h(t_0) \geq 0$, integrating (A.1) yields

$$\begin{aligned} &\Delta V(h(t_0), V_u) \\ &= \int_{t_0}^{\infty} \left[f(h) - c(\hat{\imath}) - rV_u + \lambda\hat{p}\Delta V(\hat{\phi}, V_u) - \lambda\psi\left(\hat{\phi}\right)\theta(\hat{p})\right] D(t, t_0)\,\mathrm{d}t, \end{aligned} \tag{A.5}$$

where the dependence of the integrand on t is suppressed. Given $h(t_0)$, the sequence of efficient choices from t_0 onward must maximize the right-hand side of (A.5). That is, the sequence, $\{\hat{\imath}(t), \hat{\phi}(t), \hat{p}(t)\}_{t \geq t_0}$, solves:

$$\begin{aligned} &\Delta V(h(t_0), V_u) \\ &= \max \int_{t_0}^{\infty} \left[f(h) - c(\hat{\imath}) - rV_u + \lambda\hat{p}\Delta V(\hat{\phi}, V_u) - \lambda\psi\left(\hat{\phi}\right)\theta(\hat{p})\right] D(t, t_0)\,\mathrm{d}t, \end{aligned} \tag{A.6}$$

subject to (2.1), where $h(t_0)$ is taken as given.

Denote the right-hand side of (A.6) as $(\mathcal{T}\Delta V)(h(t_0), V_u)$, where $\mathcal{T}$ is the implied mapping on ΔV. Let $\mathcal{C}$ be the space that contains all continuous functions defined on $[0, h^*] \times [0, V^*]$, with the sup-norm. Then, $\mathcal{T}$ is a self-map on $\mathcal{C}$. I prove that $\mathcal{T}$ satisfies Blackwell's sufficient conditions for a monotone contraction; i.e., it is monotone and has discounting (see Stokey et al., 1989). Thus, there is a unique fixed point of $\mathcal{T}$ in $\mathcal{C}$, which is $\Delta V(h, V_u)$.

Since $\lambda p \geq 0$, it is clear that $\mathcal{T}$ is an increasing mapping. That is, for any ΔV_A and ΔV_B in $\mathcal{C}$, with $\Delta V_A(h, V_u) \geq \Delta V_B(h, V_u)$ for all (h, V_u), $\mathcal{T}\Delta V_A(h, V_u) \geq \mathcal{T}\Delta V_B(h, V_u)$ for all (h, V_u). To verify that $\mathcal{T}$ has discounting, let $a \in (0, \infty)$ be an arbitrary constant and consider any $\Delta V \in \mathcal{C}$. Define the function $(\Delta V + a)$ by $(\Delta V + a)(h, V_u) = \Delta V(h, V_u) + a$ for all (h, V_u). The optimal choices under the value function $(\Delta V + a)$ are denoted with a subscript a, and the optimal choices under the value function ΔV are denoted without the subscript a. For any given $h(t_0)$, I have

$$\begin{aligned} &\mathcal{T}(\Delta V + a)(h(t_0), V_u) \\ = \;& \int_{t_0}^{\infty} \left[\tilde{f}(h_a, V_u) - c(\hat{\imath}_a) - rV_u + \lambda\hat{p}_a\Delta V(\hat{\phi}_a, 1) - \lambda\psi\left(\hat{\phi}_a\right)\theta(\hat{p}_a)\right] D_a(t, t_0)\,\mathrm{d}t \\ & + a\int_{t_0}^{\infty} \lambda\hat{p}_a D_a(t, t_0)\,\mathrm{d}t \\ \leq \;& \mathcal{T}\Delta V(h(t_0), V_u) + a\int_{t_0}^{\infty} \lambda\hat{p}_a D_a(t, t_0)\,\mathrm{d}t. \end{aligned}$$

The inequality follows from the definition that $\mathcal{T}\Delta V(h(t_0), V_u)$ is the maximized value of the first integral. Since $p \leq \bar{p} < \infty$, then

$$\int_{t_0}^{\infty} D_a(t, t_0)\,\mathrm{d}t \geq \int_{t_0}^{\infty} e^{-(r+\delta+\lambda\bar{p})(t-t_0)} dt = \frac{1}{r+\delta+\lambda\bar{p}}.$$

Using this result and the definition in (A.4), I derive:

$$\begin{aligned}\int_{t_0}^{\infty}\lambda\hat{p}D(t,t_0)\,\mathrm{d}t &= \int_{t_0}^{\infty}(r+\delta+\lambda\hat{p})\,D(t,t_0)\,\mathrm{d}t-(r+\delta)\int_{t_0}^{\infty}D(t,t_0)\,\mathrm{d}t\\ &= -\int_{t_0}^{\infty}\mathrm{d}D(t,t_0)-(r+\delta)\int_{t_0}^{\infty}D(t,t_0)\,\mathrm{d}t\\ &= 1-(r+\delta)\int_{t_0}^{\infty}D(t,t_0)\,\mathrm{d}t\leq\frac{\lambda\bar{p}}{r+\delta+\lambda\bar{p}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{T}\left(\tilde{V}+a\right)(h(t_0),V_u) &\leq \mathcal{T}\Delta V(h(t_0),V_u)+a\int_{t_0}^{\infty}\lambda\hat{p}_a D_a(t,t_0)\,\mathrm{d}t\\ &\leq \mathcal{T}\Delta V(h(t_0),V_u)+\frac{\lambda\bar{p}}{r+\delta+\lambda\bar{p}}a.\end{aligned}$$

Because $\lambda\bar{p}<\infty$ and $r+\delta>0$, then $\frac{\lambda\bar{p}}{r+\delta+\lambda\bar{p}}\in(0,1)$. This establishes that $\mathcal{T}$ has discounting, and so $\mathcal{T}$ has a unique fixed point $\Delta V(h,V_u)$ in $\mathcal{C}$.

For any $\Delta V\in\mathcal{C}$, it is straightforward to verify that $\mathcal{T}$ has the following properties: (i) If $\Delta V(h,V_u)$ is (weakly) increasing in h, then $\mathcal{T}\Delta V(h,V_u)$ is increasing in h, and strictly so for $h<h^*$; (ii) If $\Delta V(h,V_u)$ is (weakly) decreasing in V_u, then $\mathcal{T}\Delta V(h,V_u)$ is strictly decreasing in V_u; (iii) If $\Delta V(h,0)\geq 0$, then $\mathcal{T}\Delta V(h,0)\geq 0$, with strict inequality for $h>0$; (iv) If $\Delta V(h,V^*)\leq 0$, then $\mathcal{T}\Delta V(h,V^*)\leq 0$, with strict inequality for $h<h^*$. With the contraction mapping argument (see Stokey et al., 1989), these properties imply that the fixed point $\Delta V(h,V_u)$ is increasing in h (strictly so for $h<h^*$), strictly decreasing in V_u, satisfying $\Delta V(h,0)\geq 0$ (with strict inequality for all $h>0$), and satisfying $\Delta V(h,V^*)\leq 0$ (with strict inequality for all $h<h^*$).

The number V_u satisfies (2.6), which can be rewritten as

$$rV_u=f(h_u)+\max_{(\phi,p)}\left[p\Delta V(\phi,V_u)-\psi(\phi)\,\theta(p)\right].$$

In the maximization problem above, the choices (ϕ,p) lie in the compact set $[0,h^*]\times[0,\bar{p}]$. Since $\Delta V(\phi,V_u)$ is continuous on $[0,h^*]\times[0,V^*]$, the Theorem of the Maximum implies that the right-hand side of the above equation is continuous in V_u. It is also decreasing in V_u, because $\Delta V(\phi,V_u)$ is so. Recall that $h_u<h^*$. Since $\Delta V(h,0)\geq 0$ and $\Delta V(h,V^*)\leq 0$ for all h, the right-hand side is strictly positive at $V_u=0$ and strictly less than rV^* at $V_u=V^*$. Therefore, there is a unique $V_u\in(0,V^*)$ that satisfies the above equation.

To verify the properties of V in Lemma 3.1, return to the problem in (2.2), where the efficient choices are functions of h. Because $V(h)$ is continuous and the domain of h is compact, V is bounded. For $V(h)$ to be bounded, the right-hand side of (2.2) must be unbounded and, in particular, $\upsilon(h)$ must be bounded. Since $\frac{\mathrm{d}V(h)}{\mathrm{d}t}$ appears on the right-hand side through $\upsilon(h)$ (see (2.4)), this derivative must exist and be bounded. Note that $\frac{\mathrm{d}V(h)}{\mathrm{d}t}=V'(h(t))\,\hat{\imath}(t)$, where (2.1) is used. Because $c(\infty)=\infty$, the optimal choice i is bounded. Existence and boundedness of $\frac{\mathrm{d}V(h)}{\mathrm{d}t}$ imply that $V'(h)$ exists and is bounded.

Then, the job upgrading problem becomes (2.5). Given h, the objective function of the maximization problem in (2.5) is strictly concave in i. The optimal choice satisfies

$$V'(h) - c'(i) \leq 0 \text{ and } i \geq 0,$$

where the two inequalities hold with complementary slackness. If $i > 0$, then $V'(h) = c'(i) > 0$, as in (3.1), in which case $V(h)$ is strictly increasing. If $i = 0$, then $V'(h) \leq c'(0) = 0$. However, since $V(h)$ is (weakly) increasing, $V'(h) \geq 0$. Thus, if $i = 0$, then $V'(h) = 0 = c'(0)$. That is, (3.1) holds for both $i > 0$ and $i = 0$. Finally, whenever $\frac{\mathrm{d}i}{i\,\mathrm{d}t}$ exists, differentiating (3.1) with respect to t and using (2.1) yields (3.2). Because $c''(i) > 0$ for all $i \geq 0$, then $V''(h) < 0$ if and only if $\frac{\mathrm{d}i}{i\,\mathrm{d}t} < 0$. **QED**

B. Proof of Proposition 3.2

Differentiating (3.1) with respect to t yields $V''(h)\,i = c''(i)\,\frac{\mathrm{d}i}{\mathrm{d}t}$, where $h = h(t)$ and $i = i(h(t))$. This shows that $V''(h)\,i(h)$ exists if $\frac{\mathrm{d}i}{\mathrm{d}t}$ exists. Because $i(h)$ solves (3.1) and $c''(i) > 0$ for all $i \geq 0$, the function $i(h)$ is continuous. To prove that $\upsilon'(h)$ exists and satisfies the envelope condition for (2.5), I calculate the one-sided derivatives of υ. Let h be an arbitrary job type and $\varepsilon > 0$ an arbitrarily small number. Temporarily denote the objective function in (2.5) as $F(i,h) = V'(h)\,i - c(i)$. The derivative of υ on the left side of h is

$$\upsilon'(h_-) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left[F(i(h),h) - F(i(h-\varepsilon), h-\varepsilon)\right].$$

Because $i(h-\varepsilon)$ is the optimal when the job type is $(h-\varepsilon)$, then $F(i(h-\varepsilon), h-\varepsilon) \geq F(i(h), h-\varepsilon)$. This inequality implies

$$\upsilon'(h_-) \leq \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left[F(i(h),h) - F(i(h), h-\varepsilon)\right] = V''(h)\,i(h).$$

On the other hand, because $i(h)$ is optimal when the job type is h, then $F(i(h),h) \geq F(i(h-\varepsilon), h)$, which implies

$$\upsilon'(h_-) \geq \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\left[F(i(h-\varepsilon),h) - F(i(h-\varepsilon), h-\varepsilon)\right] = V''(h)\,i(h).$$

Notice that the last equality uses continuity of $i(h)$. Thus, $\upsilon'(h_-) = V''(h)\,i(h)$. A similar procedure proves that the derivative of υ on the right side of h is $\upsilon'(h_+) = V''(h)\,i(h)$. Thus, $\upsilon'(h)$ exists and is given as $\upsilon'(h) = V''(h)\,i(h)$.

To establish that $s'(h)$ exists, consider (2.2). The left-hand side of (2.2) is differentiable since $V'(h)$ exists. Since $f'(h)$ and $\upsilon'(h)$ exist, $s'(h)$ must exist; otherwise, the right-hand side of (2.2) would fail to be differentiable. To prove that $s'(h)$ satisfies the envelope condition for (2.3), I calculate the one-sided derivatives of s. In contrast to the proof that $v'(h)$ satisfies the envelope condition, I cannot use continuity of the policy functions in job search, since this continuity is yet to be established. Instead, I use the result that $s'(h)$ exists, which was just proven. Consider an arbitrary h and an arbitrarily small number $\varepsilon > 0$. Temporarily denote the objective function in (2.3) as $F(\phi, p, h)$. Let $(\phi(h), p(h))$ be the policy functions of the efficient choices of (ϕ, p). Because

$$F(\phi(h-\varepsilon), p(h-\varepsilon), h-\varepsilon) \geq F(\phi(h), p(h), h-\varepsilon),$$

then the left side derivative of $s(h)$ satisfies:

$$\begin{aligned} s'(h_-) &= \lim_{\varepsilon\downarrow 0} \tfrac{1}{\varepsilon} [F(\phi(h), p(h), h) - F(\phi(h-\varepsilon), p(h-\varepsilon), h-\varepsilon)] \\ &\leq \lim_{\varepsilon\downarrow 0} \tfrac{1}{\varepsilon} [F(\phi(h), p(h), h) - F(\phi(h), p(h), h-\varepsilon)] \\ &= -p(h) V'(h). \end{aligned}$$

The last expression is the partial derivative of $F(\phi, p, h)$ with respect to h. This partial derivative exists, as shown in Lemma 3.1. Similarly, because

$$F(\phi(h+\varepsilon), p(h+\varepsilon), h+\varepsilon) \geq F(\phi(h), p(h), h+\varepsilon),$$

the right side derivative of $s(h)$ satisfies:

$$\begin{aligned} s'(h_+) &= \lim_{\varepsilon\downarrow 0} \tfrac{1}{\varepsilon} [F(\phi(h+\varepsilon), p(h+\varepsilon), h+\varepsilon) - F(\phi(h), p(h), h)] \\ &\geq \lim_{\varepsilon\downarrow 0} \tfrac{1}{\varepsilon} [F(\phi(h), p(h), h+\varepsilon) - F(\phi(h), p(h), h)] \\ &= -p(h) V'(h). \end{aligned}$$

Since $s'(h)$ exists, as shown above, these results on the one-sided derivatives imply

$$-p(h) V'(h) \leq s'(h_+) = s'(h) = s'(h_-) \leq -p(h) V'(h).$$

Therefore, $s'(h) = -p(h) V'(h)$.

Differentiate the Bellman equation (2.2) with respect to V. Substituting the results for $v'(h)$ and $s'(h)$ yields (3.3). Substituting $V'(h) = c'(i)$ and $V''(h) i = c''(i) \frac{\mathrm{d}i}{\mathrm{d}t}$ yields (3.4). **QED**

C. Proof of Lemma 3.3

For (i), consider an arbitrary $h \in (0, h^*)$ such that $p(h) > 0$. Given $p > 0$, the optimal choice $\phi(h)$ must satisfy $\phi(h) > h$, because $\phi \leq h$ implies that the return on search is $-\psi(\phi)\theta(p) < 0$. Also, given p, a marginal increase in the search target ϕ increases the expected social value by

$$\Delta(\phi) \equiv V'(\phi) - \psi'(\phi) \frac{\theta(p)}{p}.$$

The optimal choice ϕ must satisfy $\phi(h) < h^*$: If $\phi(h) \geq h^*$, then $V'(\phi) = 0$ and $\Delta(\phi) < 0$, in which case the return on search can be increased by reducing the search target. Thus, $h < \phi(h) < h^*$. Since $\phi(h)$ is interior, it satisfies the first-order condition, $\Delta(\phi(h)) = 0$, which is equivalent to (3.5).

For (ii), given the efficient choice ϕ, the objective function of job search in (2.3) is strictly concave in p. Thus, the efficient choice $p(h)$ satisfies the complementary slackness condition, (3.6). Given ϕ_u, the efficient choice p_u for an unemployed worker satisfies a similar condition.

For (iii), consider the case where $p(h) > 0$. I prove that $\Delta'(\phi(h)) \geq 0$ is inconsistent with optimality, where $\Delta(\phi)$ is defined above. If $\Delta'(\phi(h)) > 0$, then $\phi(h)$ achieves a local minimum instead of a maximum, which is clearly not optimal. Suppose $\Delta'(\phi(h)) = 0$. Consider the choice $\left(\tilde{\phi}, \tilde{p}\right)$, where $\tilde{p} = p(h) + \varepsilon_p$ and $\tilde{\phi} = \phi(h) - \varepsilon_\phi$. Let $\varepsilon_p > 0$ be a sufficiently small number and $\varepsilon_\phi > \frac{\psi(\phi)p\theta''(p)\varepsilon_p}{2\psi'(\phi)[p\theta'(p)-\theta(p)]}$, where $\phi = \phi(h)$ and $p = p(h)$. Because $\theta'' > 0$ and $p\theta' > \theta$ for all $p > 0$ by assumption, then $\varepsilon_\phi > 0$, and $\varepsilon_\phi \to 0$ as $\varepsilon_p \to 0$. Since $p(h) > 0$, the first-order condition of p, (3.6), holds as equality. Thus, the change from the choice $(\phi(h), p(h))$ to $\left(\tilde{\phi}, \tilde{p}\right)$ has no first-order effect on the return on search. Since $\Delta'(\phi(h)) = 0$, the second-order effect of this change is

$$-\frac{1}{2}\psi\theta''\varepsilon_p^2 + \left(\theta' - \frac{\theta}{p}\right)\psi'\varepsilon_p\varepsilon_\phi > 0,$$

where I substituted $V' = \psi'\frac{\theta}{p}$ and used the inequality between ε_ϕ and ε_p. Since this result contradicts the optimality of $(\phi(h), p(h))$, then $\Delta'(\phi(h)) < 0$ must hold if $p(h) > 0$. If ψ is linear, this result implies $V''(\phi(h)) < 0$. **QED**

D. Proof of Proposition 3.4

Assume that $p(h)$ exists and is unique for every h. For (i) of the proposition, consider any $h \in (0, h^*)$ such that $p(h) > 0$. Define $\Phi(p)$ by $V'(\Phi(p)) = \psi'(\Phi(p))\frac{\theta(p)}{p}$. By (iii) of Lemma 3.3, $\Phi(p)$ is well-defined and single valued, with $\phi(h) = \Phi(p(h))$. Since $\frac{\theta(p)}{p}$ is strictly increasing in p, then $\Phi'(p) < 0$. Thus, $\phi'(h) > 0$ if and only if $p'(h) < 0$. To prove $p'(h) < 0$, denote the marginal contribution of the matching rate to the return on search at $\phi = \Phi(p)$ as

$$M(p, h) = V(\Phi(p)) - V(h) - \psi(\Phi(p))\theta'(p).$$

It is clear from (3.6) that $M(p(h), h) = 0$, since $p(h) > 0$ in the case considered here. Moreover, the optimality of $p(h)$ requires the return on search to be concave in p at

$p = p(h)$; i.e., the partial derivative of M with respect to p satisfies $M_1(p(h), h) \le 0$. If $M_1(p(h), h) = 0$, then for an arbitrarily small $\varepsilon > 0$,

$$M(p(h-\varepsilon), h-\varepsilon) - M(p(h), h) \approx -M_2\varepsilon = V'(h)\varepsilon > 0,$$

where I have used the fact that $V'(h) = c'(i) > 0$. This implies $M(p(h-\varepsilon), h-\varepsilon) > M(p(h), h) = 0$, which contradicts the optimality of $p(h-\varepsilon)$ when the job type is $h-\varepsilon$. Thus, $M_1(p(h), h) < 0$. Differentiating the equation $M(p(h), h) = 0$ yields

$$p'(h) = \frac{V'(h)}{M_1(p(h), h)} < 0.$$

For (ii), define $h_a = \sup\{h : p(h) > 0\}$. Because $p'(h) < 0$ for all h such that $p(h) > 0$, it is clear that $p(h) > 0$ if and only if $h < h_a$. By (3.5), h_a satisfies

$$V'(\phi(h_a)) = \psi'(\phi(h_a)) \lim_{p\downarrow 0} \frac{\theta(p)}{p} = \psi'(\phi(h_a))\theta'(0).$$

With h_c defined by (3.7), this result shows $h_c = \phi(h_a)$. Since $p(h) > 0$ for all $h < h_a$, the first-order condition for $p(h)$ in (3.6) is satisfied with equality for all $h < h_a$. In the limit $h \uparrow h_a$, this first-order condition becomes:

$$V(h_c) - V(h_a) - \psi(h_c)\theta'(0) = 0.$$

With h_T defined by (3.8), this equation implies $h_a = h_T$ and so $h_c = \phi(h_T)$. Because $\theta'(0) > 0$ by assumption, (3.7) implies $V'(h_c) > 0 = V'(h^*)$, and so, $h_c < h^*$. Similarly, (3.8) implies $V(h_c) > V(h_T)$, and so $h_c > h_T$.

For (iii), I prove that $\frac{\mathrm{d}i(h)}{\mathrm{d}t} < 0$ for all $h \in [h_T, h^*)$. This result further implies $V''(h) < 0$ for all $h \in [h_T, h^*)$, because $\frac{\mathrm{d}i}{\mathrm{d}t} < 0$ if and only if $V(h)$ is strictly concave (see Lemma 3.1). For all $h \in [h_T, h^*)$, $p(h) = 0$, and so (3.4) yields

$$\frac{\mathrm{d}i}{\mathrm{d}t} = \frac{1}{c''(i)}\left[(r+\delta)c'(i) - f'(h)\right].$$

This equation and (2.1) form a system of differential equations for (i, h). Define

$$I(h) = c'^{-1}\left(\frac{f'(h)}{r+\delta}\right).$$

If $i(h) < I(h)$ for all $h \in [h_T, h^*)$, then $\frac{\mathrm{d}i}{\mathrm{d}t} < 0$ for all such h. Suppose, to the contrary, that $i(h(t_0)) \ge I(h(t_0))$ for some $t_0 < \infty$ and $h(t_0) \in [h_T, h^*)$. Then $\frac{\mathrm{d}i(h(t))}{\mathrm{d}t}|_{t=t_0} \ge 0$ and $\frac{\mathrm{d}h(t)}{\mathrm{d}t}|_{t=t_0} = i(h(t_0)) \ge I(h(t_0)) > 0$, where the last strict inequality comes from $h(t_0) < h^*$. Re-use the notation $\hat{\imath}(t) = i(h(t))$. For a sufficiently small $\varepsilon > 0$, $\hat{\imath}(t_0+\varepsilon) \ge \hat{\imath}(t_0)$ and $h(t_0+\varepsilon) > h(t_0)$, where the inequality on h is strict because $\frac{\mathrm{d}h(t)}{\mathrm{d}t}|_{t=t_0} > 0$. Then,

$$c'(\hat{\imath}(t_0+\varepsilon)) \ge c'(\hat{\imath}(t_0)) \ge \frac{f'(h(t_0))}{r+\delta} > \frac{f'(h(t_0+\varepsilon))}{r+\delta}.$$

The first inequality comes from $\hat{\imath}(t_0+\varepsilon) \geq \hat{\imath}(t_0)$ and convexity of c. The second inequality comes from the hypothesis $\hat{\imath}(t_0) \geq I(h(t_0))$. The third (strict) inequality comes from $h(t_0) < h(t_0+\varepsilon)$ and strict concavity of f. This result implies $\hat{\imath}(t_0+\varepsilon) > I(h(t_0+\varepsilon))$, which in turn implies $\frac{\mathrm{d}\hat{\imath}(t)}{\mathrm{d}t}|_{t=t_0+\varepsilon} > 0$ and $\frac{\mathrm{d}h(t)}{\mathrm{d}t}|_{t=t_0+\varepsilon} = \hat{\imath}(t_0+\varepsilon) > 0$. By induction, $\hat{\imath}(t) > I(h(t))$, $\frac{\mathrm{d}\hat{\imath}(t)}{\mathrm{d}t} > 0$ and $\frac{\mathrm{d}h(t)}{\mathrm{d}t} > 0$ for all $t \in (t_0, \infty)$. Since $\hat{\imath}(t)$ increases in t, then $\lim_{t\to\infty} \frac{\mathrm{d}h(t)}{\mathrm{d}t} = \lim_{t\to\infty} \hat{\imath}(t) \geq \hat{\imath}(t_0) \geq I(h(t_0)) > 0$. From $h(t_0)$, the path of the job type will surpass h^* in a finite length of time and will keep increasing thereafter. Because $f'(h) < 0$ for all $h < h^*$, this path cannot be socially efficient. Therefore, $\hat{\imath}(t) < I(h(t))$ and so $\frac{\mathrm{d}\hat{\imath}(t)}{\mathrm{d}t} < 0$ for all $h(t) \in [h_T, h^*)$. Similarly, $\hat{\imath}(t) > 0$ for all $h(t) \in [h_T, h^*)$. The convergence of (i, h) to the final state $(0, h^*)$ is asymptotic.

For (iv), first consider the case where ψ is linear. Then, (iii) of Lemma 3.3 implies that $V(\phi(h))$ is strictly concave in ϕ whenever $p(h) > 0$. Be the definition of h_T, $p(h) > 0$ for all $h < h_T$. Because $\phi(h_u) = \phi_u$ and $\phi(h_T) = h_c$, then $V(h)$ is strictly concave for all $h \in [\phi_u, h_c)$. With this result, (3.1) implies $c'(i) = V'(h) > V'(h_c) > 0$ and, hence, $i(h) > 0$ for all $h \in [\phi_u, h_c)$. Furthermore, (3.2) implies $\frac{\mathrm{d}i(h)}{\mathrm{d}t} = \frac{V''(h)i(h)}{c''(i(h))} < 0$ for all $h \in [\phi_u, h_c)$. Because $h_c > h_T$, these results clearly hold for $h \in [\phi_u, h_T]$.

If ψ is sufficiently convex, it is possible that $V''(h) > 0$ for some $h \in [\phi_u, h_T]$, and so $\frac{\mathrm{d}i(h)}{\mathrm{d}t} > 0$ for such h. This completes the proof of Proposition 3.4.

The remainder of this proof specifies a sufficient condition for $p(h)$ to be unique for every h. Because $p(h) = 0$ for all $h \geq h_T$, it suffices to consider only $h < h_T$. The proof of Proposition 3.4 above has shown that $p(h)$ is given by the solution to $M(p(h), h) = 0$, where $M_1(p(h), h) < 0$. The proof of Proposition 3.4 has shown that $p(h)$ is given by the solution to $M(p(h), h) = 0$, where $M_1(p(h), h) < 0$. For any $h < h_T$, the sufficient condition for $p(h)$ to exist is $M(0, h) > 0 > M(\tilde{p}, h)$ for some $\tilde{p} \geq 0$. Recall the function $\Phi(p)$ defined in the proof of Proposition 3.4. The definition of h_c in (3.7) implies $\Phi(0) = h_c$. Thus, $M(0, h) > 0$ if and only if $V(h) < V(h_c) - \psi(h_c)\theta'(0)$. This condition is equivalent to $h < h_T$. Define $\tilde{p}$ by $\Phi(\tilde{p}) = h$ so that $\tilde{p} = \Phi^{-1}(h) \geq 0$. Then, $M(\tilde{p}, h) = -\psi(h)\theta'(\tilde{p}) < 0$.

Now turn to uniqueness of $p(h)$. Since $M(p(h), h) = 0$ and $M_1(p(h), h) < 0$, the necessary and sufficient condition for $p(h)$ to be single valued is that $M_1(p, h) < 0$ for all p such that $M(p, h) = 0$. Let p be a solution to $M(p, h) = 0$. Then,

$$\Phi'(p) = \frac{-(\theta' - \theta/p)\psi'}{\psi''\theta - V''p},$$

where the argument in ψ and V is $\Phi(p)$, which is suppressed. Substituting this derivative and $V' = \psi'\theta/p$, I can calculate

$$M_1(p, h) = \frac{[(\theta' - \theta/p)\psi']^2}{\psi''\theta - V''p} - \psi\theta''.$$

The possibility of $V'' > 0$ makes it difficult to obtain the general condition for $M_1(p,h) < 0$ for all p such that $M(p,h) = 0$. If $V(h)$ is (weakly) concave for all $h \geq \phi_u$, I can provide sufficient conditions for $p(h)$ to be unique. If $V'' \leq 0$, then

$$M_1(p,h) \leq \frac{[(\theta' - \theta/p)\psi']^2}{\psi''\theta} - \psi\theta''.$$

Then, a sufficient condition for $M_1(p,h) < 0$ is

$$\frac{\psi\psi''}{(\psi')^2} > \frac{(\theta' - \theta/p)^2}{\theta\theta''}.$$

Using $P(\theta)$ as the inverse of $\theta(p)$, I can write this condition as

$$\frac{\psi\psi''}{(\psi')^2} > \frac{(1 - \theta P'/P)^2}{-\theta P''/P'}. \tag{D.1}$$

If V is strictly concave, the above condition can be relaxed to a weak inequality.

Consider the example $\psi(h) = \psi_0 h^{\psi_1}$, where $\psi_0 > 0$ and $\psi_1 \geq 1$, and the matching functions in Example 2.1. With the urn-ball matching function, (D.1) becomes

$$1 - \frac{1}{\psi_1} > \frac{e^{1/\theta} - 1 - \frac{1}{\theta}}{(e^{1/\theta} - 1)^2}.$$

It can be verified that the right-hand side is a decreasing function of $1/\theta$ for all $1/\theta \geq 0$, and so its maximum is achieved at $1/\theta \to 0$, which is $1/2$. Thus, a sufficient condition for $p(h)$ to be unique under the urn-ball matching function is $\psi_1 \geq 2$. If the matching function is the generalized telephone matching function in Example 2.1, then (D.1) becomes

$$1 - \frac{1}{\psi_1} > \left\{(1+\rho)\left[\left(\frac{p_0}{\theta}\right)^\rho + 1\right]\right\}^{-1}.$$

Since the right-hand side of this inequality is maximized at $\theta \to \infty$, a sufficient condition for $p(h)$ to be unique is $1 - \frac{1}{\psi_1} \geq \frac{1}{1+\rho}$, i.e., $\psi_1 \geq 1 + \frac{1}{\rho}$. **QED**

E. Proof of Proposition 4.1

Assume that the joint density $\omega(h,t)$ exists for all $(h,t) \in [\phi_u, h^*] \times [0,\infty)$. Recall that for any tenure $t \geq 0$, the job type $h(t)$ is reached by a worker who stays in the same firm all the way to t ever since being employed. If a worker succeeds in moving to another job, the job type reached by the worker at actual tenure t is higher than $h(t)$. Since $h(t)$ is the lowest job type reached by a worker with actual tenure t, then $\omega\left(\tilde{h}, t\right) = 0$ for all $\tilde{h} < h(t)$. Similarly, for any $h \in [\phi_u, h^*]$, the longest tenure needed to reach h is $t(h)$, where $t(h)$ is the inverse function of $h(t)$. Thus, $\omega\left(h, \tilde{t}\right) = 0$ for all $\tilde{t} > t(h)$.

To characterize the joint distribution of employed workers, consider employed workers in $(h,h^*]\times\{t\}$, i.e., the group of employed workers whose job types are higher than h and whose tenure is t. The density of these workers is $-\bar{\Omega}_t(h,t)$. In a small interval of time Δ, all of these workers exit the group because they either separate which resets tenure to zero, or stay in the same firm which increases tenure to $t+\Delta$. Thus, the outflow of workers from the group is $-\bar{\Omega}_t(h,t)$. The inflow is the group of workers employed in $L\cup(h,h^*]$ and $t-\Delta$ who do not separate from their jobs, where $L=\{\tilde{h}\le h:\tilde{h}+i\left(\tilde{h}\right)\Delta>h\}$. For these workers, the passing of time increases tenure to t. If their job types were in L, which is outside $(h,h^*]$, job upgrading increases their job types into $(h,h^*]$. If their job types were already in $(h,h^*]$, they remain in this interval. The density of this inflow of workers is

$$\begin{aligned}
& \textstyle\int_{\tilde{h}\in L\cup(h,h^*]}\omega\left(\tilde{h},t-\Delta\right)\left[1-\delta\Delta-\lambda p\left(\tilde{h}\right)\Delta\right]\mathrm{d}\tilde{h} \\
= \; & -\bar{\Omega}_t(h,t-\Delta)+\textstyle\int_{\tilde{h}\in L}\omega\left(\tilde{h},t-\Delta\right)\mathrm{d}\tilde{h} \\
& -\Delta\textstyle\int_{\tilde{h}\in L\cup(h,h^*]}\omega\left(\tilde{h},t-\Delta\right)\left[\delta+\lambda p\left(\tilde{h}\right)\right]\mathrm{d}\tilde{h}.
\end{aligned}$$

Notice that p is inside the integral because the endogenous separation rate depends on $\tilde{h}$.

The measure of the set L in the domain of h goes to 0 in the limit $\Delta\to 0$. Moreover, the ratio of the measure of L to Δ goes to $i(h)$ in the limit $\Delta\to 0$. Thus,

$$\lim_{\Delta\to 0}\frac{1}{\Delta}\int_{\tilde{h}\in L}\omega\left(\tilde{h},t-\Delta\right)\mathrm{d}\tilde{h}=i(h)\,\omega(h,t)\,.$$

Equating the inflow to the outflow from the group $(h,h^*]\times\{t\}$, dividing by Δ and taking $\Delta\to 0$, I get:

$$i(h)\,\omega(h,t)+\bar{\Omega}_{tt}(h,t)=\int_h^{h^*}\left[\delta+\lambda p\left(\tilde{h}\right)\right]\omega\left(\tilde{h},t\right)\mathrm{d}\tilde{h}. \tag{E.1}$$

Since $\omega(h,t)=\bar{\Omega}_{ht}(h,t)$ and $\bar{\Omega}_h(h,\infty)=\bar{\Omega}_t(h,\infty)=\bar{\Omega}(h,\infty)=0$, integrating (E.1) over tenure in (t,∞) yields (4.3).

The marginal distribution of employed workers over h is $G(h)=1-\bar{\Omega}(h,0)$, and the corresponding density is $g(h)=-\bar{\Omega}_h(h,0)$. Setting $t=0$ in (4.3) yields (4.4). Similarly, since the lowest employed job type is ϕ_u, the marginal distribution of employed workers over tenure t is $G_t(t)=1-\bar{\Omega}(\phi_u,t)$, and the corresponding density is $g_t(t)=-\bar{\Omega}_t(\phi_u,t)$. Because all workers employed at $h=\phi_u$ just came out of unemployment, their actual tenure is 0. Thus, $\omega(\phi_u,t)=0$ for all $t>0$, and so $\bar{\Omega}_h(\phi_u,t)=0$ for all $t\ge 0$.[19] Using this fact and setting $h=\phi_u$ in (4.3), I get (4.5) for all $t\ge 0$.

Differentiating (4.5) with respect to t yields:

$$g_t'(t)=-\delta g_t(t)-\int_{\phi_u}^{h^*}\lambda p\left(\tilde{h}\right)\omega\left(\tilde{h},t\right)\mathrm{d}\tilde{h}.$$

[19] This is an implication of the earlier result that $\omega\left(\tilde{h},t\right)=0$ for all $\tilde{h}<h(t)$, because $h(t)>h(0)=\phi_u$ for all $t>0$.

Because $\omega(h, t)$ exists for all (h, t), this equation shows that $g_t'(t)$ exists and $g_t'(t) < 0$ for all $t \in [0, \infty)$, as stated in (i). Similarly, differentiating (4.3) with respect to h yields:

$$\frac{\mathrm{d}}{\mathrm{d}h}[i(h)g(h)] = -[\delta + \lambda p(h)]g(h) + \omega(h, 0).$$

Thus, $[i(h)g(h)]$ is differentiable for all h. Since this derivative is also equal to $i'(h)g(h) + i(h)g'(h)$, then $g'(h)$ exists if and only if $i'(h)$ exists and $i(h) \neq 0$. By Proposition 3.4, $i(h) > 0$ for all $h < h^*$. Note that $i'(h) = \frac{\mathrm{d}i(h)}{i(h)\mathrm{d}t}$. Because $\frac{\mathrm{d}i(h)}{\mathrm{d}t}$ exists for all $t \in [0, \infty)$ (see (3.4)), then $i'(h)$ exists if and only if $i(h) \neq 0$, i.e., if and only if $h < h^*$. Thus, $g'(h)$ exists for all $h \in [\phi_u, h^*)$. This result is (ii). Finally, to verify (iii), consider any $h \in (h_c, h^*)$. Since h_c is the highest job type that can be reached by job switching, an employed worker who has $h > h_c$ must have experienced job upgrading immediately before reaching such a job type and, hence, must have strictly positive tenure. That is, $\omega(h, 0) = 0$ for all $h \in (h_c, h^*)$. Because $p(h) = 0$ for these job types, then $\frac{\mathrm{d}}{\mathrm{d}h}[i(h)g(h)] = -\delta g(h) < 0$. This result also shows that $g'(h) < 0$ if and only if $-i'(h) < \delta$. Since $i'(h) = \frac{\mathrm{d}i(h)}{i(h)\mathrm{d}t}$ and $p(h) = 0$, (3.4) implies that $g'(h) < 0$ if and only if $f'(h) < \delta i c''(i) + (r + \delta)c'(i)$ where $i = i(h)$. **QED**

F. Procedures of Calibration and Computation

F.1. Calibration Procedure

The model is calibrated at the monthly frequency. The discount rate r is determined by the target, $(1 + r)^3 = 1.0125$. The parameter α in the output function is interpreted as the labor share in output and set as $\alpha = 0.64$. Since $f'(h^*) = 0$ and $f'(h) = \alpha h^{\alpha-1} - f_0$, then $f_0 = \alpha(h^*)^{\alpha-1}$, where $h^* = 100$ is normalization. The calibration target on home production yields $f(h_u) = 0.36 f(h^*)$, which determines h_u. The monthly transition rate from employment to unemployment is $\delta = 0.026$, which is taken from the Current Population Survey (CPS). The value $\lambda = 1$ is chosen as the benchmark. With the target $u = 0.065$, the steady state yields $p_u = \delta(1 - u)/u = 0.374$. Because $\varepsilon_u \equiv \frac{\mathrm{d}\ln p_u}{\mathrm{d}\ln \theta(p_u)} = 1 - (p_u)^\rho$, the target $\varepsilon_u = 0.39$ yields $\rho = 0.5$. The targets, $f(\phi_u)/f(h^*) = 0.45$ and $\psi(\phi_u)/f(\phi_u) = 0.1$, imply $\phi_u = 9.63$ and $\psi_0 = 0.309 \times 9.63^{-\psi_1}$. Once ψ_1 is determined, ψ_0 is determined. The remaining parameters, (ψ_1, p_0, c_1), minimize the distance *drest*, where

$$drest\ (\%) \equiv \left[(p_u - 0.374)^2 + \left(\frac{\phi_u}{h^*} - 0.0963\right)^2\right]^{1/2} \times 100\%.$$

The job finding rate instead of the unemployment rate is used in the distance measure in order to increase the sensitivity of the distance to the parameters that affect the job-finding rate. When ψ_1 varies, ψ_0 varies according to the relationship determined above: $\psi_0 = 0.309 \times 9.63^{-\psi_1}$. Grid search is used to solve the above problem.

To construct the grid of (ψ_1, p_0, c_1), I approximate the optimality conditions of (ϕ_u, p_u) to get the pre-estimates of the parameters. Start with the envelope condition of ϕ_u:

$$V'(\phi_u) = \frac{f'(\phi_u) + V''(\phi_u)\, i(\phi_u)}{r + \delta + \lambda p(\phi_u)}.$$

Ignoring the effects of (V''', i', p') on V'', I get

$$V''(\phi_u) \approx \frac{f''(\phi_u)}{r + \delta + \lambda p(\phi_u)}.$$

This approximation is likely to over-estimate $[-V''(\phi_u)]$ by ignoring $i' < 0$ and $p' < 0$. Substituting this approximation into the envelope condition of ϕ_u yields:

$$V'(\phi_u) \approx \frac{f'(\phi_u)}{r + \delta + \lambda p(\phi_u)} + \frac{f''(\phi_u)\, i(\phi_u)}{[r + \delta + \lambda p(\phi_u)]^2}.$$

The first order condition of ϕ_u implies

$$\theta(p_u) \approx \frac{p_u}{\psi'(\phi_u)} V'(\phi_u).$$

Suppose that $V_u = V(h_0)$ for some h_0. Approximate $[V(h_0) - V(\phi_u)]$ near ϕ_u to the second order:

$$V(\phi_u) - V(h_0) \approx V'(\phi_u)(\phi_u - h_0) - \frac{V''(\phi_u)}{2}(\phi_u - h_0)^2.$$

Substituting this approximation into the first order condition of p_u and dividing the result by the first order condition of ϕ_u, I get:

$$1 - \frac{h_0}{\phi_u} - \frac{\phi_u V''(\phi_u)}{2V'(\phi_u)}\left(1 - \frac{h_0}{\phi_u}\right)^2 \approx \frac{\psi(\phi_u)}{\phi_u \psi'(\phi_u)} \frac{\theta'(p_u)}{p_u \theta(p_u)}.$$

With the functional form of ψ in the calibration, $\frac{\psi(h)}{h\psi'(h)} = \frac{1}{\psi_1}$ for all h. Also, $\frac{\theta'(p_u)}{p_u\theta(p_u)} = \frac{1}{\varepsilon_u}$, which was calculated above. Thus, the above approximation yields

$$\psi_1 \approx \frac{1}{\varepsilon_u}\left[1 - \frac{h_0}{\phi_u} - \frac{\phi_u V''(\phi_u)}{2V'(\phi_u)}\left(1 - \frac{h_0}{\phi_u}\right)^2\right]^{-1}.$$

To use these formulas, I need the values of h_0, $i(\phi_u)$ and $p(\phi_u)$, which are endogenous. From the definition of h_0, $V(h_0) = V_u$. Note that the equivalent job type of an unemployed worker is h_u, where $f(h_u)$ is an unemployed worker's home production. An employed worker has the opportunity of getting the job type upgraded but an unemployed worker does not. This difference is likely to induce $V(h_0) < V(h_u)$. I set $h_0 = \frac{2}{3\lambda} h_u$. Also, set $i(\phi_u) = 0.25\phi_u$ and $\lambda p(\phi_u) = 0.80 p_u$ in the above approximation. Then, the above approximation yields the pre-estimate $\psi_1 = 2.6862$. The first-order condition of ϕ_u yields the approximation of $\theta(p_u)$. The matching function implies the pre-estimate of p_0 as $p_0 = \theta(p_u)\left[(p_u)^{-\rho} - 1\right]^{1/\rho} = 0.4462$. The first-order condition of i yields the pre-estimate of c_1 as $c_1 = V'(\phi_u) / [2i(\phi_u)] = 0.0529$, where the above approximations for $V'(\phi_u)$ and $i(\phi_u) = 0.25\phi_u$ are used.

A grid is chosen for each parameter in (ψ_1, p_0, c_1) around the pre-estimates. The distance *drest* is minimized at $p_u = 0.3792$ and $\phi_u = 9.5884$, and the minimum is 0.523%. The values of (ψ_1, p_0, c_1) that minimize *drest* are reported in Table 1. Figure F.1 depicts the distance *drest* over the grid of (ψ_1, p_0) in the upper panel and of (p_0, c_1) in the lower panel. Notice that ψ_1 and p_0 are identified well, since moving away from the identified values increases *drest* significantly. The parameter c_1 is also identified well, despite that the distance *drest* depends on c_1 non-monotonically.

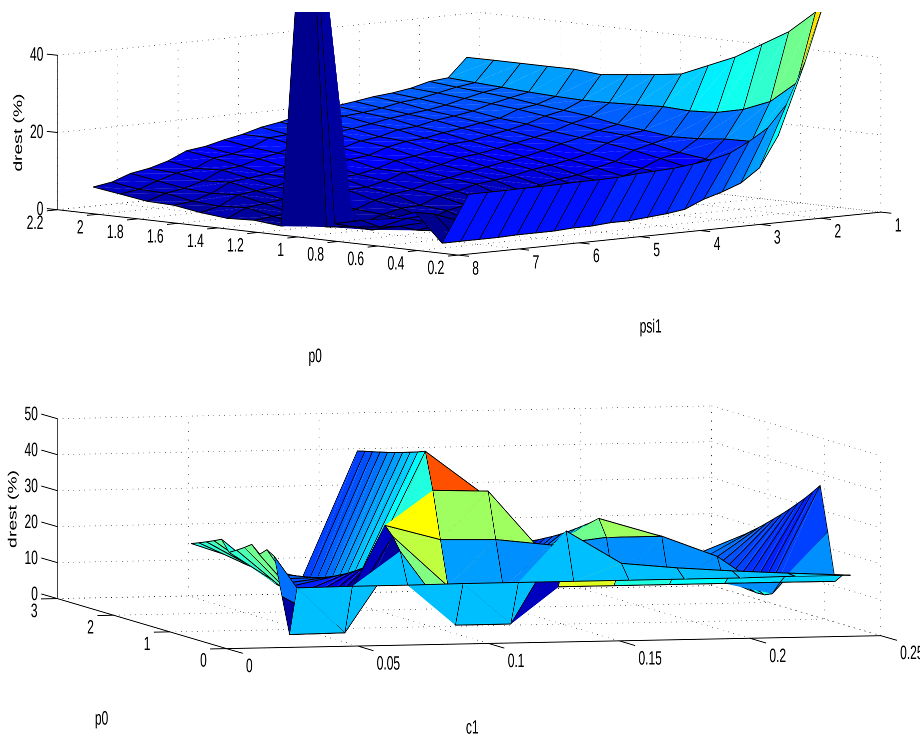

Figure F.1 The distance between $(p_u, \phi_u/h^*)$ and their targets

F.2. Computation Procedure

The following procedure computes the efficient allocation first and then the distributions:

Step 1. Specify the parameter values, the forms of exogenous functions, and the domain of h. Compute h^*. Construct the Chebyshev basis for the value function.

Step 2. Compute the social value function and its derivative with Chebyshev projection. The same step yields the policy functions of efficient job upgrading and job search. Use grid search to solve the maximization problems in (2)-(4) below.

(1) Specify an initial function $V(h)$ and a number V_u. If $V(h)$ is not weakly increasing, modify it so. Compute the derivative $V'(h)$.

(2) Solve the efficient job upgrading problem (2.5) for the policy function $i(h)$ and the implied return on upgrading $\upsilon(h)$.

(3) Solve the efficient job search problem (2.3) for the policy functions $(\phi(h), p(h))$ and the implied return on search $s(h)$.

(4) Solve the efficient job search problem of an unemployed worker for the efficient allocation (ϕ_u, p_u) and the implied value of unemployment V_u.

(5) Update the function $V(h)$ by (2.2) and the number V_u by (2.6). Repeat (1)-(4) until convergence.

Step 3. Compute h_T and h_c. Use Chebyshev projection to spline the functions $V(h)$, $i(h)$, $\phi(h)$ and $p(h)$. Ensure $i(h) \geq 0$, $\phi(h) \geq h$, $p(h) \geq 0$, and $p(h) = 0$ for all $h \geq h_T$. Also compute the projection for the function $\frac{1}{i(h)}$, to be used later. Compute V' and ensure $V'(h) \geq 0$.

Step 4. Compute (h, i, ϕ, p) as functions of tenure, i.e., $\left(h(t), \hat{\imath}(t), \hat{\phi}(t), \hat{p}(t)\right)$. The function $h(t)$ is given by the differential equation $\frac{\mathrm{d}h(t)}{\mathrm{d}t} = i(h(t))$. However, numerically integrating this equation can be complicated. Instead, compute the inverse of $h(t)$ which can can be approximated well by using the differential equation: $t(h) = \frac{1}{i(h)}$. Since $t(\phi_u) = 0$, then

$$t(h) = \int_{\phi_u}^{h} \frac{1}{i(z)} \mathrm{d}z.$$

Since the projection for the function $\frac{1}{i(h)}$ was computed above, one can directly integrate the basis in the projection. Numerically invert $t(h)$ to get $h(t)$. Note that there is no iteration here. Once the function $h(t)$ is computed, evaluating the splines of $i(h)$, $\phi(h)$ and $p(h)$ on $h(t)$ yields:

$$\hat{\imath}(t) = i(h(t)), \; \hat{\phi}(t) = \phi(h(t)), \; \hat{p}(t) = p(h(t)).$$

Step 5. Compute the steady state distribution. The partial differential equation, (4.3), characterizes the distribution of employed workers over (h, t). Rather than solving this equation numerically, which yields significant imprecision, I use the policy functions to simulate the joint distribution by the Monte Carlo method. The procedure is as follows:

(1) Discretize the values of the unemployment duration as $A = \{0, \mathrm{d}t, 2\mathrm{d}t, ..., t_M \mathrm{d}t\}$, where $\mathrm{d}t > 0$ is a small number and t_M is a large integer. Tenure is also discretized as

set A. Job types are discretized as the set $h(A) = \{h_0, h_1, h_2, ..., h_M\}$, where $h_0 = \phi_u$ and $h_j = h(j\mathrm{dt})$.

(2) Set the number of individuals to be simulated as a large number N, say, $N = 10^6$. Divide them into unemployed $N_u = n_u N$ and employed $N_e = (1 - n_u) N$. Specify the initial distribution of employed workers over the grid $A \times h(A)$ and the initial distribution of unemployed workers over the grid A. Use these initial distributions as the cumulative average of the distributions across simulations.

(3) Take each worker from the cumulative average distribution across previous simulations and simulate the worker's transition in one period. For each worker employed at $(h_k, j\mathrm{d}t)$, make a random draw on job switching according to the probability $\lambda p(h_k)\mathrm{d}t$ and a random draw of exogenous separation according to the probability $\delta\mathrm{d}t$. If the worker separates into unemployment, set the unemployment duration to 0. If the worker switches the job, set the worker's tenure to 0 and the new job type to the level on the grid $h(A)$ that is closest to $\phi(h_k)$. If the worker stays in the firm, increase the worker's tenure to $(j+1)\mathrm{d}t$ and the job type to h_{k+1}. Similarly, for an unemployed worker, make a random draw on job finding according to the probability $p_u\mathrm{d}t$. If the worker finds a job, set the worker's tenure to 0 and the job type to $h_0 = \phi_u$.

(4) Calculate the distribution of employed workers over $A \times h(A)$ and the distribution of unemployed workers over A.

(5) Combine the newly simulated distributions in (4) with the previous cumulative average distribution of workers to update the cumulative average distribution of workers across simulations.

(6) Repeat (3)-(5) until convergence of the cumulative average distribution of workers.

(7) With the distribution obtained in (6), compute the distribution of employed workers over (h, t) and unemployed workers over the unemployment duration. Compute other statistics from the distribution.

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