# Meetings and Mechanisms*

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**Abstract**

We analyze a market in which sellers compete for heterogeneous buyers by posting mechanisms. A general meeting technology governs how buyers and sellers meet. We introduce a one-to-one transformation of this meeting technology that helps to clarify and extend many of the existing results in the literature, which has focused on two special cases: urn-ball and bilateral meetings. We show that the optimal mechanism for sellers is to post auctions combined with a reserve price equal to their own valuation and an appropriate fee (or subsidy) which is paid by (or to) all buyers meeting the seller. Under weak conditions, there exists a unique equilibrium. Even when there are (congestion) externalities in the meeting process, the equilibrium is efficient. Finally, we analyze the sorting patterns between heterogeneous buyers and sellers and show under which conditions high-value sellers attract high-value buyers.

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# 1 Introduction

One of the fundamental questions in the economics literature concerns the choice of a trading mechanism by a seller who wishes to sell a good. A large literature in mechanism design analyzes this question in the context of a monopolistic seller and often finds that auctions dominate prices. Recent work by Eeckhout and Kircher (2010b) however points out that in a market setting in which sellers compete for buyers with private valuations by posting mechanisms, the process that governs meetings between buyers and sellers—i.e., the *meeting technology*—is crucially important for the choice of mechanism. If buyers randomly select a seller and sellers are unconstrained in the number of buyers that they can meet (urn-ball meetings), then auctions are indeed useful instruments to identify the buyer with the highest valuation. The efficient equilibrium in this case consists of a single market in which all sellers post auctions and all buyer types pool, as this maximally spreads high-type buyers across sellers. However, if sellers are, for example, time-constrained and can only meet one buyer at a time (bilateral meetings), low-type buyers may crowd out high-type buyers. In that case, sellers prefer to post prices, which induces perfect separation of buyers into homogeneous submarkets. Although the assumption of either bilateral or urn-ball meetings is nearly universal in the search literature[1], neither technology is necessarily an adequate description of real-life markets. In many cases, it might be more realistic to consider a technology which allows a seller to meet and learn the type of multiple but not all buyers who are interested in matching with him.

In this paper we allow for general meeting technologies and study how they affect posted mechanisms. Specifically, we analyze an environment in which a continuum of buyers and sellers try to trade subject to the frictions generated by an arbitrary meeting technology. We allow for all sorts of positive and negative spillovers in the meeting process. The only constraint we impose is that all buyers who visit a submarket (consisting of sellers that post a particular mechanism and the buyers who search in this market) are equally likely to *meet* with a seller in that submarket. Of course, the likelihood of being *matched* will depend on the buyer's valuation. For simplicity, we focus on constant-returns-to-scale meeting functions (so we can focus on the buyer-seller ratio rather than having to keep track of both buyers and sellers in a submarket). Our main finding is that in the decentralized equilibrium, each seller cannot do better than post a second-price auction, combined with a meeting fee to be paid by (or to) each buyer who meets the seller. Intuitively, in a large market, sellers take buyers' equilibrium payoffs as given, making sellers the residual claimant on any extra surplus that they create and providing them with an incentive to post efficient mechanisms. Auctions

[1] Bilateral meetings can be found in e.g. Moen (1997), Guerrieri et al. (2010), and Menzio and Shi (2011). The urn-ball is used in e.g. Peters (1997), Burdett et al. (2001), Shimer (2005), and Albrecht et al. (2014). Levin and Smith (1994) considers a single auction with a random number of buyers.

guarantee that the good is allocated efficiently, while the meeting fees price any positive or negative externalities in the meeting process, providing all agents with a payoff equal to their social contribution. This also implies that the decentralized market equilibrium is efficient. The efficiency result does arise not because there are no externalities and agency costs but because they offset each other.

We explain this below. Note that buyers have private information about their valuations but that this only affects the posted mechanism and not their payoffs. Also, in a large market, the total surplus equals the sum of buyers' and seller's marginal contribution to surplus. Consider first a market with a finite number of buyers and sellers with an invariant meeting technology like the urn-ball (no meeting externalities are present). Suppose a new seller enters the market and draws $m$ buyers away from the other sellers. So in the absence of this new seller, the buyers would have created value elsewhere. Albrecht et al. (2014) label this a business stealing externality. The expected total value of the $m$ buyers visiting the new seller equals the difference between the highest and second highest valuation which is a standard information rent. In a large market, the following holds: (i) the new entrant takes buyers' market utility as given, (ii) the probability that two or more of the $m$ buyers come from the same seller is zero, (iii) if you add or remove a buyer from an existing auction, the sum of the seller and other buyer's payoffs remains the same, see Albrecht et al. (2014).[2] This implies that a buyer's expected marginal contribution to the surplus equals his expected payoff. Note that the $m$ buyers should have the same value between visiting the new seller and the other sellers and therefore, in a large market, the buyer's payoff is the same as the business stealing externality. This is the main intuition for why sellers receive their marginal contribution (or equivalently why entry is efficient).[3] Now return to general meeting technologies. This case is more complicated because now a buyer can also impose positive or negative meeting externalities on other buyers. Why could a decentralized market still reach the efficient outcome in this case? We show that all those externalities can be priced with an appropriate entrance fee/subsidy. The buyers' payoff now reflects not only an informational rent but also a fee or subsidy to stimulate or discourage the buyer to contact the seller in question. In other words, the decrease in the surplus associated with the new entrant in the rest of the market *by definition* is the marginal contribution that each of the $m$ buyers would have

[2] Either the new buyer did not have the highest valuation than he is irrelevant or he has the highest valuation and he receives his valuation minus the highest valuation across the original $n$ buyers (i.e., he receives his marginal contribution to the surplus).

[3] In a small market, there is a positive probability that two of the $m$ buyers would have ended in the same auction had the new seller not entered. By attracting these two buyers simultaneously, the agency cost might be smaller than the business stealing externality which may result in excessive entry. For example, suppose that two of the buyers with valuations: 0.5 and 0.8 would in the absence of the new entrant have gone to a seller with one buyer with valuation: 0.2. The payoff of the winning buyer is 0.3 (agency cost) while their joint contribution to the surplus is 0.6 (business stealing externality).

created at the sellers they would have gone to had the seller in question not entered. That value consists of a direct effect if they would have the highest valuation and an indirect effect consisting of possible search externalities. Furthermore, the buyers' payoff is always equal to their marginal contribution to the surplus. Thus in a large market, the business stealing externality plus the sum of the meeting externalities equals the agency cost (buyers' value) and this explains why despite the presence of coordination frictions and private information, the problem reduces to that of a competitive market for queues and that consequently, the efficiency of seller entry carries over to general meeting technologies.

Why is it important to allow for general meeting technologies? Below, we give a number of examples of different environments with different meeting technologies which sometimes are neither bilateral nor urnball. For example, in the labor market, screening costs prevent firms from learning the type of all their applicants.[4] In the housing market, viewings are often costly for the sellers and this may affect the meeting rate (sellers may want to meet with only a subset of the buyers or only organize a viewing if there are at least $i$ and at most $j$ buyers).[5] In the marriage market, meetings typically used to be bilateral but since the introduction of online dating, many-to-one meetings are now also common. In the market for goods and services, procurers can typically meet multiple contractors per period while car dealers typically meet with one buyer at a time.[6] For products sold online, meetings are sometimes many to one (eBay) and sometimes bilateral (Amazon). [7] Finally, schools often require a minimum number of students before a new class is opened or some boat companies require a minimum number of customers before they operate. In such an environment, buyers can impose positive spillovers on each other.[8] All those different environments give rise to different meeting technologies which in turn affect the sellers' choices of mechanisms. If sellers do not know the exact meeting technology, it is useful to derive a class of meeting technologies for which, for example, an auction without fee is the optimal mechanism. Our general approach allows for this. The strongest case for why the meeting technology matters for the mechanisms we observe comes from recent technological changes in terms of the digitization of the labor market. Agrawal et al. (2015) report that platforms like Upwork (previously oDesk) or Freelancer enable employers from high-income countries to outsource tasks to contractors from mainly low-income countries. The number of hours worked at oDesk increased with 55% from 2011 to 2012, with the 2012 total wage bill just over 360 million dollar. A 2014 New York Times article states: "*It's also helping to raise the standard*

[4]See Fraja and Sákovics (2001), Lester and Wolthoff (2014), and Wolthoff (2016).

[5]Albrecht et al. (2014) and Lester and Wolthoff (2014) allow for many-to-one meetings in the housing market.

[6]See for example Kim and Kircher (2012).

[7]Amazon has a huge storage capacity so typically, all buyers will be served and it is as if buyers meet bilaterally with the products.

[8]See for example Geromichalos (2012)

*of living for workers in developing countries. The rise of these marketplaces will increase global productivity by encouraging better matching between employers and employees.*"[9]. The new online platforms facilitate many-on-one meetings (also for small firms) and by doing so, they allow for different wage mechanisms. For example, on oDesk, firms posted jobs and contractors could apply to those jobs by submitting cover letters and bids to those job postings, see Agrawal et al. (2015). A bid indicates the amount a contractor is willing to be paid to work on a job. Employers have the option to interview and negotiate over bids with applicants before hiring. This nicely illustrates how a new technology affects the meeting technology and that firms respond by adjusting the wage mechanism accordingly.

There exists a small literature that considers general meeting technologies. Eeckhout and Kircher (2010b) were the first who emphasized that the meeting technology affects the mechanisms that are offered in equilibrium, but they derive the equilibrium mechanism only for a subset of technologies. Moreover, they consider two buyer types while we allow for general distributions of buyer valuations. Lester et al. (2015) provide a full characterization of the equilibrium, but in a simpler environment in which all buyers are ex ante identical. Cai et al. (2016) apply the tools that are developed in this paper to derive conditions on the meeting technology for which the equilibrium features either perfect separation or perfect pooling of different types of buyers, and they relate those conditions to other properties of meeting technologies that have been derived in the literature, like invariance (Lester et al., 2015) and non-rivalry (Eeckhout and Kircher, 2010b).

Finally, Einav et al. (2013) and Backus et al. (2015) give evidence that on eBay, price posting has become increasingly more popular than auctions and the latter estimated that the hassle cost of the auction (discount relative to price posting) has increased from 4 to 8%. In order to highlight the role of meeting technologies, we abstract from hassle cost here. Even without those costs, price posting is sometimes the most efficient mechanism. In particular when a standardized good is sold and there is no rationing so that each buyer is served with probability one. If, however, a unique product is sold and many-on-one meetings are possible, our model predicts that auctions should be used as selling mechanism. This is consistent with what we observe for the selling of unique or collector's goods like houses, watches, jewelry and classic cars. Here many-on-one meetings are possible and the selling mechanism is an auction (possibly with fees). [10].

Our main methodological innovation is that in order to analyze these general meeting technologies, we introduce a new function, $\phi$. This new function is an alternative repre-

[9]See `http://www.nytimes.com/2014/02/16/business/small-business-joining-a-parade-of-outsourcing.html`

[10]For example, Catawiki (http://www.catawiki.com/) and Sothebys use auctions. The latter charges entry fees for buyers that differ per auction. For housing auctions in the UK, it is also common that all participating buyers pay a fee.

sentation of the meeting technology and it specifies the probability for a seller to meet at least one buyer from a given subset. This function depends on the buyer-seller ratio and the fraction of buyers in this subset. We show that the expected surplus at a seller only depends on the integral of this function over buyer valuations, $x$ (where the relevant subset equals the expected number of buyers with a valuation above $x$). In a seminal paper, Myerson (1981) formulated and solved the "optimal auction design problem" among all possible ways of selling the good, by introducing the virtual valuation function. His tools have been widely applied in auction theory and other areas where private information plays a key role. We use $\phi$ to solve the optimal auction design problem in a competitive environment for any meeting technology. It turns out that there is a simple relation between $\phi$ and the virtual valuation. In standard auction theory, the seller's payoff is obtained by integrating a buyer's virtual valuation against the distribution of highest valuations. We show here that for a given buyer-seller ratio, $\lambda$, the latter term can simply be replaced by $1 - \phi(\lambda(1 - F(x)), \lambda)$, which simply equals the probability that no buyers arrive with valuations above $x$. So the introduction of $\phi$ adds a lot of generality to the competing mechanism literature at relatively low cost.

Finally, we show that when sellers are also heterogeneous our existence, uniqueness and efficiency results carry through. In this environment, sorting issues arise and a natural question is under which conditions do we get positive assortative matching (PAM), i.e. high valuation buyers visit on average higher valuation sellers.

After describing this environment in detail in section 2 and the alternative representation of the meeting technology in section 3, we start our analysis in section 4 by considering the trade-off of a social planner between the desire to spread high-type buyers as much as possible and the risk of them being crowded out by low-type buyers. Whether or not it is desirable that all buyers or sellers are active in the market depends on whether the derivative of $\phi$ with respect to the buyer-seller ratio is positive or negative. We also show in this section that if there are $n$ buyer types, it is optimal to open no more than $n + 1$ sub markets. Section 5 how the planner's solution can be decentralized. Finally, section 6 deals with sorting issues and the proofs are relegated to the appendix.

# 2 Model

Before we provide a precise description of the details of the model, we give a brief overview of the problem here. The model is static and has two stages. First, sellers post a selling mechanism, and then after observing all selling mechanisms, buyers decide which seller to visit, subject to meeting frictions. Our main objectives are to derive which selling mechanisms will be preferred in equilibrium, to characterize the allocation of buyers across sellers, and to establish whether or not the decentralized equilibrium is efficient.

**Agents and Preferences.** The economy consists of a measure 1 of sellers, indexed by $j \in [0,1]$, and a measure $\Lambda > 0$ of buyers. Both buyers and sellers are risk-neutral. Each seller possesses a single unit of an indivisible good, for which each buyer has unit demand. Initially, we will assume that all sellers have the same valuation for their good, which we normalize to zero; later, in Section 6, we will consider seller heterogeneity. Buyers have a valuation between 0 and 1, and the buyer value distribution is denoted by $G(x)$ with $0 \leq x \leq 1$ and $G(0) < 1$. Buyers' valuations are private information.

**Mechanisms.** In the first stage, each seller posts and commits to a direct anonymous mechanism to attract buyers. The mechanism specifies, for each buyer $i$, a probability of trade and an expected payment as a function of: (i) the total number $n$ of buyers that successfully meet with the seller; (ii) the valuation $x_i$ that buyer $i$ reports; and (iii) the valuations $x_{-i}$ reported by the $n-1$ other buyers.[11] Anonymity is reflected by the fact that the mechanism does not condition the allocation and payoffs on the identity of buyers.

**Search.** We refer to all identical mechanisms as a *submarket*. After observing all submarkets, each buyer chooses the one in which he wishes to attempt to match. Because we consider a large market, we assume that buyers can not coordinate their visiting strategies, such that buyers must use symmetric strategies in equilibrium; this is a standard assumption in the literature (see e.g. Montgomery, 1991; Burdett et al., 2001; Shimer, 2005).

**Meeting Technology.** Consider a submarket with a measure $b$ of buyers and a measure $s$ of sellers. The meetings between buyers and sellers are frictional and governed by a *meeting technology*, which we model analogous to Eeckhout and Kircher (2010b). The meeting technology is anonymous; it treats all buyers (sellers) in a symmetric way, i.e., independent of their identity. A buyer can meet at most one seller, while a seller may meet multiple buyers. Define $\lambda = b/s$ as the *queue length* in this submarket.[12] The probability of a seller meeting $n$ buyers, $n = 0, 1, 2, \ldots$, is given by $P_n(\lambda)$, which is assumed to be continuously differentiable. Because each buyer can meet at most one seller, $\sum_{n=1}^{\infty} nP_n(\lambda) \leq \lambda$. By an accounting identity, the probability for a buyer to be part of an $n$-to-1 meeting is $Q_n(\lambda) \equiv nP_n(\lambda)/\lambda$ with $n \geq 1$. Finally, the probability that a buyer fails to meet any seller is $Q_0(\lambda) \equiv 1 - \sum_{n=1}^{\infty} Q_n(\lambda)$.

[11]In line with most of the literature, we abstract from mechanisms that condition on other mechanisms present in the market. See Epstein and Peters (1999) and Peters (2001) for a detailed discussion.

[12]For simplicity, we assume here that a positive measure of buyers and sellers visit the submarket. This need not be the case; e.g. the economy could have a continuum of submarkets with each a measure zero of buyers and sellers. In that case, we could use Radon-Nykodym derivatives to define queue lengths.

**Strategies.** Let $M$ be the set of all direct anonymous mechanisms equipped with some natural $\sigma$-algebra $\mathcal{M}$. A seller's strategy is a probability measure $\delta_s$. A buyer needs to decide on whether or not to participate in the market, and if yes, which sellers (who are characterized by the mechanisms they post) to visit. To acknowledge that a buyer's strategy depends (only) on his value $x$ and the fact that—due to the lack of coordination—buyers treat all sellers who post the same mechanism symmetrically, we denote his strategy by $\delta_b(x, \cdot)$, a measure on $(M, \mathcal{M})$. If $\delta_b(x, M) < 1$, then buyers with value $x$ will choose not to participate in the market with probability $1 - \delta_b(x, M)$, in which case their payoff will be zero.[13] The allocation of all buyers with value less or equal to $x$ across posted mechanisms can be formally denoted as a measure $\Psi(x, \cdot)$ on $\mathcal{M}$. Individual strategies and the aggregate allocation satisfy, for any measurable subset $B$ of $\mathcal{M}$,

$$\Psi(x, B) = \int_0^x \delta_b(y, B) dG(y).$$

Since a buyer can only visit a mechanism if a seller posted it, we require that for each $x$, the measure $\Psi(x, \cdot)$ is absolutely continuous with respect to $\delta_s$.[14] The Radon-Nikodym derivative $d\Psi(x, \cdot)/d\delta_s$ determines the queue length and queue composition—i.e., how many buyers and what types of buyers—for each mechanism (almost surely) in the support of $\delta_s$. Formally, for (almost every) mechanism $\omega$ in the support of $\delta_s$, the queue length $\lambda(\omega)$ and queue composition $F(x, \omega)$ are given by

$$\lambda(\omega) F(x, \omega) = \frac{d\Psi(x, \cdot)}{d\delta_s}. \tag{1}$$

**Payoffs.** Note that for any mechanism $\omega \in M$, the expected payoff of a seller who posts mechanism $\omega$ is completely determined by $\omega$ and its queue length $\lambda(\omega)$ and queue composition $F(x, \omega)$. Therefore, we can denote it by $R(\omega, \lambda(\omega), F(x, \omega))$. Similarly, let $V(z, \omega, \lambda(\omega), F(x, \omega))$ denote the expected payoff of a buyer with value $z$ from visiting a submarket with mechanism $\omega$ which has queue length $\lambda(\omega)$ and queue composition $F(x, \omega)$.[15]

**Market Utility and Beliefs.** We now define conditions on buyers' and sellers' strategy $(\delta_s, \delta_b)$ which need to be satisfied in equilibrium. First, consider the optimality of buyers' strategies. The *market utility function* $U(z)$ is defined to be the maximum utility that a

[13] We assume that sellers always post a selling mechanism. This is without loss of generality, since sellers can stay inactive by posting a sufficiently inattractive selling mechanism, e.g. a reserve price above 1.

[14] This rules out the scenario in which a zero measure of sellers attracts a positive measure of buyers. This restriction is natural and can be justified by the optimal choices of buyers and sellers (see below).

[15] $R(\omega, \lambda(\omega), F(x, \omega))$ can be calculated as $\sum_{n=1}^{\infty} P_n(\lambda) R_n(\omega, F(x, \omega))$, where $R_n(\omega, F(x, \omega))$ denotes the expected payoff of the seller when $n$ buyers arrive. $V(z, \omega, \lambda(\omega), F(x, \omega))$ can be calculated in a similar way.

buyer with value $z$ can obtain by visiting a seller or being inactive.

$$U(z) = \max\left(\max_{\omega \in \text{supp}(\delta_s)} V(z, \omega, \lambda(\omega), F(x, \omega)), 0\right). \tag{2}$$

where $\lambda(\omega)$ and $F(x,\omega)$ follow from equation (1). Of course, optimality of buyers' choices requires that buyers choose the mechanism that yields the highest payoff. Formally, we have

$$V(x, \omega, \lambda(\omega), F(x, \omega)) \leq U(x) \quad \text{with equality if } \omega \text{ is in the support of } \delta_b(x, \cdot). \tag{3}$$

Next we consider the optimality of sellers' strategies. All posted mechanisms should generate the same expected payoff $\pi$ and there should be no profitable deviations. That is,

$$R(\omega, \lambda(\omega), F(x, \omega)) \leq \pi \quad \text{with equality if } \omega \text{ is in the support of } \delta_s \tag{4}$$

Furthermore, there should be no profitable deviation. A seller considering a deviation to a mechanism $\widetilde{\omega}$ not in the support of $\delta_s$ needs to form beliefs regarding the queue $(\lambda(\widetilde{\omega}), F(x, \widetilde{\omega}))$ that he will be able to attract. We call a queue $(\lambda, F(x))$ *compatible* with the mechanism $\widetilde{w}$ and the market utility function $U(x)$ if for any $z$,

$$V(z, \widetilde{\omega}, \lambda, F(x)) \leq U(z) \quad \text{with equality if } z \text{ is in the support of } F(x). \tag{5}$$

Of course, for any mechanism $\omega$ in the support of $\delta_s$, $(\lambda(\omega), F(x, \omega))$ is compatible with mechanism $\omega$ and the market utility function because of the optimal search behavior of buyers. The literature usually assumes that when posting $\widetilde{w}$, the seller will expect the most favorable queue among all queues that are compatible with $\widetilde{\omega}$ and the market utility function (see, for example, McAfee, 1993; Eeckhout and Kircher, 2010a,b). We will show that—with some mild restrictions on the meeting technology—this assumption is unnecessary: when $\widetilde{\omega}$ is (without loss of generality) an auction with entry fee, these restrictions imply that there is only one possible queue compatible with $\widetilde{\omega}$ and the market utility function.

**Equilibrium Definition.** We can now define an equilibrium as follows.

**Definition 1.** *A directed search equilibrium is a pair $(\delta_s, \delta_b)$ of strategies with the following properties:*

1. *Each $\widetilde{\omega}$ in the support of $\delta_s$ maximizes $R(\omega, \lambda(\omega), F(x, \omega))$, where, depending on whether or not $\omega$ belongs to the support of $\delta_s$, $\lambda(\omega)$ and $F(x, \omega)$ are given by equations (1) and (5), respectively.*

2. *For each buyer type $z$, every $\widetilde{\omega}$ in the support of $\delta_b(z)$ maximizes $V(z, w, \lambda(\omega), F(x, \omega))$.*

*If for any mechanism $\omega$ in the support of $\delta_s$ the buyer value $V(z, \omega, \lambda(\omega), F(x, \omega))$ is negative, then buyers with value $z$ will choose inactivity and $\delta_b(z, M) = 0$.*

3. *Aggregating queues across sellers does not exceed the total measure of buyers of each type.*

# 3 Alternative Representation

In this section, we present a transformation of the meeting technology that greatly simplifies the analysis. In particular, we introduce a new function $\phi(\mu, \lambda)$ with $0 \le \mu \le \lambda$, which is defined as

$$\phi(\mu, \lambda) = 1 - \sum_{n=0}^{\infty} P_n(\lambda) \left(1 - \frac{\mu}{\lambda}\right)^n . \tag{6}$$

To understand this function, consider a submarket in which sellers face queues of length $\lambda$. Suppose now that a fraction $\mu/\lambda$ of the buyers in the submarket has an arbitrary characteristic, e.g. we color them "blue." Since the meeting technology treats different buyers symmetrically, $\phi(\mu, \lambda)$ then represents the probability that a seller meets at least one blue buyer.

In many situations, by choosing "blue buyers" as buyers with valuations above some level, the function $\phi$ allows us to study competing mechanisms with general meeting functions in a way that is both more tractable and more intuitive than with $P_n(\lambda)$, $n = 0, 1, \ldots$. The following proposition establishes that the function $\phi$ is an equivalent way of characterizing frictions in the market. That is, no information is lost by considering $\phi$ instead of $P_n$.

**Proposition 1.** *There is a one-to-one relationship between $\phi(\mu, \lambda)$ and $P_n(\lambda)$, $n = 0, 1, 2, \ldots$.*

*Proof.* See appendix A.1. □

To develop intuition for $\phi$, consider a submarket in which a measure $\mu$ of buyers has high valuations, while the remaining measure $\lambda - \mu$ has low valuations. If $\Delta\lambda$ more buyers visit this submarket, then the probability that the seller meets at least one *incumbent* high-value buyer becomes $\phi(\mu, \lambda + \Delta\lambda)$. Therefore, $\phi_\lambda(\mu, \lambda)$ measures the effect of the new entrants on the meeting probabilities between sellers and incumbent high-value buyers: $\phi_\lambda(\mu, \lambda) < 0$ (resp. $> 0$) represents negative (resp. positive) meeting externalities. In the special case of $\phi_\lambda(\mu, \lambda) = 0$, there is no meeting externalities among buyers.

For future reference, note that

$$\phi_\mu(\mu, \lambda) = \sum_{n=1}^{\infty} Q_n(\lambda) \left(1 - \frac{\mu}{\lambda}\right)^{n-1} . \tag{7}$$

That is, $\phi_\mu(\mu, \lambda)$ is the probability for a buyer to be part of a meeting in which all other buyers (if any) have low valuations. In this case, the buyer increases social surplus directly, since the good would have been allocated to a low-value buyer in his absence. It is easy to see that $\phi_\mu(\mu, \lambda)$ is decreasing in $\mu$, implying that $\phi(\mu, \lambda)$ is concave in $\mu$, which holds strictly if and only if $P_0(\lambda) + P_1(\lambda) < 1$.[16] Two special cases are worth mentioning: i) $\phi_\mu(0, \lambda) = 1 - Q_0(\lambda)$, i.e. the probability that a buyer meets a seller, and ii) $\phi_\mu(\lambda, \lambda) = Q_1(\lambda)$, i.e. the probability that a buyer meets a seller without other buyers.

**Examples of Meeting Technologies.**

1. *Bilateral.* With bilateral meeting technologies, each seller meets at most one buyer, i.e., $P_0(\lambda) + P_1(\lambda) = 1$ with $P_1(\lambda)$ strictly concave. In this case, $\phi(\mu, \lambda) = P_1(\lambda)\,\mu/\lambda$.[17]

2. *Invariant.* Invariant meeting technologies are defined by the absence of meeting externalities, i.e. $\phi_\lambda(\mu, \lambda) = 0$ for any $0 \le \mu \le \lambda$.[18] One example is the urn-ball meeting technology, which specifies that the number of buyers meeting a seller follows a Poisson distribution with a mean equal to the queue length $\lambda$. That is, $P_n(\lambda) = e^{-\lambda}\lambda^n/n!$, which yields $\phi(\mu, \lambda) = 1 - e^{-\mu}$.

3. *Non-Rival.* Eeckhout and Kircher (2010b) define a meeting technology to be non-rival if $Q_0(\lambda) = q$, where $q$ is a constant, i.e., the probability that a buyer successfully meets a seller is not affected by the presence of other buyers. From equation (7), we can see that non-rival meeting technologies can also be defined by the condition $\phi_\mu(0, \lambda) = q$ for any $\lambda$, or equivalently $\phi_{\mu\lambda}(0, \lambda) = 0$.

   Note that non-rival meeting technologies are very general because any meeting technology can be approximated arbitrarily closely by a non-rival meeting technology in the following sense. Start with any meeting technology, e.g. the bilateral technology. If some buyers fail to meet sellers, we let them meet with an arbitrary small measure of sellers who were set aside initially according to a non-rival meeting technology, like urn-ball. The meeting technology obtained from the above two-stage process is non-rival since every buyer will meet a seller for sure ($Q_0(\lambda) = 0$). By making the measure of sellers in the second step close to zero, the resulting technology can be made arbitrarily close to the original one, while remaining non-rival.[19]

[16] For each $n \ge 0$, $-(1 - \mu/\lambda)^n$ is increasing and concave in $\mu$, and it is strictly concave in $\mu$ if and only if $n \ge 2$. Therefore, $\phi(\mu, \lambda)$ is strictly concave in $\mu$ if and only if there exists at least one $n \ge 2$ such that $P_n(\lambda) > 0$.

[17] To keep the exposition concise, we omit the (straightforward) derivation of $\phi(\mu, \lambda)$ for each example.

[18] Lester et al. (2015) first introduced invariant meeting technologies in terms of $P_n(\lambda)$. Cai et al. (2016) show that their definition is equivalent to $\phi_\lambda(\mu, \lambda) = 0$.

[19] Cai et al. (2016) introduce an example of such a technology.

# 4 Planner's Problem

Given the above environment, the problem of a social planner consists of two parts. First, the planner must allocate buyers and sellers to submarkets. That is, he must determine the queue length and composition (i.e., the buyer value distribution) for each seller. Second, the planner must specify the allocation of the good after meetings have taken place. We focus on the first part below, since the second part is trivial: the planner will always allocate the good to the buyer with the highest value.

**Surplus.** We start by deriving total surplus and agents' marginal contribution to this surplus in a submarket with queue length $\lambda$ and a queue composition $F(x)$. Proposition 2 presents the results.

**Proposition 2.** *Consider a submarket with a measure 1 of sellers and a measure $\lambda$ of buyers whose values are distributed according to $F(x)$. Define $\mu(x) = \lambda(1 - F(x))$. Total surplus then equals*

$$S_0 = \int_0^1 \phi(\mu(x), \lambda)dx. \tag{8}$$

*The marginal contribution to surplus by a buyer with valuation $x$ equals*

$$T(x) = \int_0^1 \phi_\lambda(\mu(z), \lambda)dz + \int_0^x \phi_\mu(\mu(z), \lambda)dz. \tag{9}$$

*A seller's marginal contribution to surplus equals*

$$R = \int_0^1 \phi(\mu(x), \lambda) - \mu(x)\phi_\mu(\mu(x), \lambda) - \lambda\phi_\lambda(\mu(x), \lambda)\, dx. \tag{10}$$

*Proof.* See appendix A.2. □

The second term of $T(x)$ reflects a buyer's direct contribution to surplus when he has the highest value in an $n$-to-1 meeting, i.e., the difference between the highest and the second highest buyer values. The first term of $T(x)$ represents positive or negative search externalities that the buyer may impose on other buyers; it does not depend on $x$, because the meeting friction treats all buyers symmetrically. In particular, if a buyer makes it easier for other buyers to meet a seller ($\phi_\lambda \geq 0$), he increases total surplus through a positive meeting externality, even if he does not have the highest value among these buyers. Similar logic applies to a negative meeting externality ($\phi_\lambda \leq 0$). Finally, since total surplus exhibits constant returns to scale, Euler's homogeneous function theorem implies that a seller's marginal contribution equals $R = S(\lambda, F) - \lambda \int_0^1 T(x)dF(x)$.

**Participation.** The above expressions allow us to now address the planner's participation decisions. In particular, the following lemma characterizes under which conditions the planner wants either all buyers or all sellers to be active.

**Lemma 1.** *If $\phi_\lambda(\mu, \lambda) \geq 0$ ($\leq 0$ resp.) for all $0 < \mu < \lambda$, then the planner will require all buyers (sellers resp.) to be active in the market.*

*Proof.* See appendix A.3. □

Intuitively, as long as buyers do not negatively affect the meeting rate of other buyers, they should be included in the market. In contrast, if they do negatively affect other buyers, then the planner will include as many sellers as possible in order to mitigate this negative externality.

**Allocation.** Next, we consider the allocation of buyers to different submarkets. To simplify notation and deliver an upper bound on the number of submarkets, assume that the number of different buyer types is finite. To be precise, suppose that there are $n$ buyer types with values $x_1, x_2, \ldots, x_n$, satisfying $x_1 < x_2 < \cdots < x_n$, and measures $\Lambda_1, \Lambda_2, \ldots, \Lambda_n$, respectively.

Consider now a submarket in which there is a positive measure of sellers, such that the queue length is well defined. Let the queue in this submarket be $(\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i)$, where $\lambda_j^i$ is the number of buyers with value $x_j$ per seller. Then, by proposition 2, total surplus per seller in this submarket can be written as

$$S_0(\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i) = \sum_{j=1}^{n} (x_j - x_{j-1}) \phi(\lambda_j^i + \cdots + \lambda_n^i, \lambda_1^i + \ldots + \lambda_n^i), \tag{11}$$

where $x_0 = 0$. To understand equation (11), start from $n = 1$. In this case, all buyers are homogeneous and a surplus of $x_1$ is generated whenever a seller meets at least one buyer, i.e. surplus is simply $x_1 \phi(\lambda_1^i, \lambda_1^i)$. When $n = 2$ and some buyers have a higher value $x_2$, the additional surplus is $x_2 - x_1$. This surplus is realized when sellers meet at least one buyer with value $x_2$. Hence, total surplus is $x_1 \phi(\lambda_1^i + \lambda_2^i, \lambda_1^i + \lambda_2^i) + (x_2 - x_1)\phi(\lambda_2^i, \lambda_1^i + \lambda_2^i)$. For general $n$, the interpretation is the same.

Suppose now that the planner creates $k$ submarkets with positive seller measures $\alpha_1, \ldots, \alpha_k$, respectively, and potentially an additional submarket with no sellers but only buyers. Of course, this additional submarket generates no surplus but could play a role in reducing possible meeting externalities. Aggregate social surplus is

$$S = \sum_{i=1}^{k} \alpha_i S_0(\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i) \tag{12}$$

subject to the standard accounting constraint

$$\sum_{i=1}^{k} \alpha_i = 1, \tag{13}$$

for sellers, and

$$\sum_{i=1}^{k} \alpha_i \lambda_j^i \leq \Lambda_j. \tag{14}$$

for each buyer type $j = 1, 2, \ldots, n$. Note that in equation (14) we have an inequality rather than an equality as in equation (13). The reason is that the planner may require some buyers not to visit any seller and thus be inactive.[20]

We define an *idle* submarket as a market that either contains only buyers or only sellers and an *active* submarket as a market where both buyers and sellers are present. Of course, the planner will never prefer coexistence of two idle markets, one for buyers and one for sellers. The following proposition limits the number of submarkets.

**Proposition 3.** *The planner's problem can be solved by opening* $n+1$ *submarkets, including one potentially idle submarket.*

*Proof.* See appendix A.4. □

The intuition behind proposition 3 is the following. By equation (12), total surplus is a convex combination of the surpluses generated by individual submarkets. The planner chooses the number of submarkets to find the maximum value that such convex combinations can reach, which simply corresponds to finding the concave hull of the individual submarket surplus function $\mathcal{S}_0$ as presented in equation (11). As a result of this correspondence, the Fenchel-Bunt Theorem provides an upper bound for the number of submarkets needed to solve the planner's problem.[21]

**Illustration.** As an illustration, consider the simple case in which all buyers are homogeneous and have value 1, $P_0(\lambda) = e^{-\lambda^2/2}$, and $P_1(\lambda) = 1 - P_0(\lambda)$. It is easy to see that $1 - P_0(\lambda)$ is convex when $\lambda \leq 1$ and concave when $\lambda \geq 1$, as the blue solid line in figure 1

[20] In the $k$ submarkets with positive seller measure, the queue length is finite and well defined. In contrast, the queue length is infinite and not formally defined in a submarket with only buyers and no sellers.

[21] The classical Caratheodory theory states that any point in the convex hull of a set $A \subset \mathbb{R}^n$ can be represented as a convex combination of $n+1$ points of $A$. Since the graph of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a subset in $\mathbb{R}^{n+1}$, the Caratheodory Theorem implies that we only need $n+2$ points from $f$ to construct its convex hull. The Fenchel-Bunt Theorem states that if the domain of $f$ is connected, then we only need $n+1$ points from $f$ for constructing its concave hull.

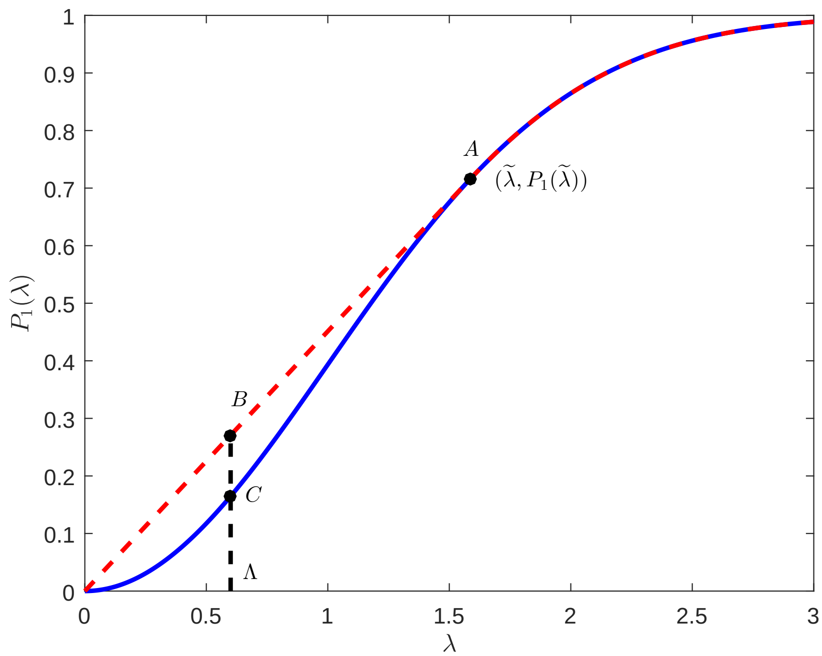


Figure 1: Illustration of Proposition 3

indicates. The concave hull of the function $1 - P_0(\lambda)$ is the red dashed line, which consists of two parts: a line segment between the origin and $(\widetilde{\lambda}, 1 - P_0(\widetilde{\lambda}))$ and the original function $1 - P_0(\lambda)$ from $\widetilde{\lambda}$ onwards. The point $\widetilde{\lambda}$ is characterized by the condition that the slopes of the original line and the tangent line of the function $1 - P_0(\lambda)$ are equal at $\widetilde{\lambda}$. Proposition 3 says that any point on the red dashed line is a convex combination of two points on the blue line. If the total buyer measure equals $\Lambda < \widetilde{\lambda}$, then the optimal allocation is point $B$ instead of point $C$, which implies that the planner will keep $1 - \Lambda/\widetilde{\lambda}$ sellers inactive (i.e. create an idle submarket) and send the buyers and the rest of the sellers to a single submarket, where the queue length is $\widetilde{\lambda}$. If the total buyer measure equals $\Lambda \geq \widetilde{\lambda}$, then the optimal allocation is to simply assign all buyers and sellers to the same submarket.

**Characterization.** Although proposition 3 shows that the planner can maximize the social surplus by opening no more than $n + 1$ submarkets, it provides no characterization of how queues will vary across submarkets. To address this question, we will show below that the planner's solution can be decentralized by sellers posting an auction with an entry fee, and we will characterize how queues of different submarkets vary with respect to the entry fee.

# 5 Decentralized Market Equilibrium

In this section, we show that the solution to the planner's problem coincides with a directed search equilibrium in which sellers compete with mechanisms. No seller can do better than posting a second-price auction combined with a meeting fee to be paid by each buyer meeting him. A negative meeting fee means that the seller pays a meeting subsidy *to* each buyer.

## 5.1 Incentive Compatibility and Payoffs

Before analyzing which mechanism sellers wish to post, we derive agents' expected payoffs. While doing this, it becomes clear how helpful our new representation of meeting technologies, $\phi$, is; despite being much more general, the analysis remains almost as simple as that of a monopolistic auction.

**Payoffs in a Monopolistic Auction.** When a monopolistic seller offers a selling mechanism, incentive compatibility requires that buyers' expected utility is intimately connected with their trading probabilities (see Myerson, 1981; Riley and Samuelson, 1981). To see this, consider $n$ buyers who participate in an efficient mechanism—i.e., a mechanism in which the buyer with the highest value trades if and only if his valuation exceeds that of the seller, like a second-price auction with no reserve price but potentially an entry fee. The expected payoff $V_n(x)$ for a buyer with value $x$ from participating in the mechanism equals

$$V_n(x) = V_n(0) + \int_0^x F^{n-1}(z)dz, \tag{15}$$

where $F^{n-1}(z)$ represents the probability that all $n-1$ other buyers have a value below $z$. Buyers' payoff is increasing and convex in their type $x$, since $F^{n-1}(z)$ is increasing in $z$. Furthermore, the seller's payoff $\pi_n$ can be written as

$$\pi_n = -nV_n(0) + \int_0^1 \left(x - \frac{1-F(x)}{f(x)}\right) dF^n(x), \tag{16}$$

where $x - (1-F(x))/f(x)$ is the virtual valuation function (Myerson, 1981) and $F^n(x)$ is the distribution of the highest valuation among $n$ buyers.

**Payoffs Under Competing Mechanisms.** The function $\phi$ allows us to derive similar results in an environment with competing mechanisms and general meeting technologies. We do this in two steps. First, we prove that the market utility function must be convex and closely related to buyers' trading probabilities. Subsequently, we derive agents' payoffs in a particular submarket and show that they resemble equations (15) and (16).

For the first step, denote the set of sellers that buyers of type $x$ visit in equilibrium by $S(x)$, pick an arbitrary $s(x) \in S(x)$ and denote by $p(x, s(x))$ the probability that a buyer of type $x$ trades when visiting seller $s(x)$. Of course, if buyers of type $x$ choose to be inactive, then we set $s(x) = \emptyset$ and $p(x, \emptyset) = 0$. The following proposition then establishes the properties of the market utility function.

**Proposition 4.** *For any set of mechanisms posted by sellers,* $p(x, s(x))$ *is non-decreasing and the market utility function* $U(x)$ *is convex, satisfying*

$$U(x) = U(0) + \int_0^x p(y, s(y))dy.$$

*If* $U(x)$ *is differentiable at point* $x_0$*, then* $p(x_0, s_0)$ *is the same for every* $s_0 \in S(x_0)$*, i.e., the probability that a buyer of type* $x_0$ *trades is the same at each seller that he may visit.*

*Proof.* See appendix A.5. □

There are several statements in proposition 4 but the basic ideas are the same as in the single seller case: (i) because of the incentive compatibility constraint, high-valuation buyers must have a higher chance of obtaining the object, and (ii) buyers' payoff is determined solely by the trading probabilities. The combination of both ideas makes the market utility function convex. As we will see later, this has important consequences for a seller's optimal choice of selling mechanism; in particular, for sellers who face a convex market utility function, the optimal selling mechanism is to post an auction with an entry fee.

The additional feature introduced by competition between sellers is that buyers also need to consider where other buyers will visit. One consequence of this is that if a buyer $x$ mixes over several submarkets, then the probabilities of winning the object in all these submarkets must be the same, for almost all buyer types $x \in [0, 1]$ (with respect to the usual Lebesgue measure). For example, suppose that both buyers of type $x$ and $x + \Delta x$ visit sellers $i$ and $j$. Since buyers' utility is an integral of trading probabilities, $U(x + \Delta x) - U(x) = p(x, i)\Delta x = p(x, j)\Delta x$. Therefore, the trading probabilities of buyers of type $x$ should be equal across the different submarkets that they visit.

For the second step, consider a submarket in which sellers post an efficient mechanism. Suppose the submarket attracts a queue $\lambda$ of buyers whose values are distributed according to $F(x)$. The following lemma then establishes agents' expected payoffs in this submarket.

**Lemma 2.** *Consider a submarket with an efficient mechanism, a queue length* $\lambda$*, and a buyer value distribution* $F(x)$*. The expected payoff for a buyer with valuation* $x$ *visiting this* *submarket is*

$$V(x) = V(0) + \int_0^x \phi_\mu(\lambda(1 - F(z)), \lambda)dz. \tag{17}$$

*The expected payoff of a seller in the submarket is*

$$\pi = -\lambda V(0) + \int_0^1 \left(x - \frac{1 - F(x)}{f(x)}\right) d\big(1 - \phi(\lambda(1 - F(x)), \lambda)\big). \tag{18}$$

*Furthermore, the set* $\{x \,|\, V(x) = U(x)\}$ *is always an interval.*

*Proof.* See appendix A.6. □

The interpretation of equation (17) is similar to equation (15). By equation (7), the term $\phi_\mu(\lambda(1 - F(z)), \lambda)$ in equation (17) is the probability that a buyer with valuation $z$ meets a seller and has the highest valuation among all buyers who arrived at the seller. Hence, for efficient mechanisms, it is simply the trading probability of the buyer. On the seller side, equation (18) is similar to equation (16). In a standard auction with $n$ bidders, a seller's expected payoff equals the virtual valuation function integrated against the distribution of the highest valuation among $n$ buyers, which is simply $F^n(x)$. In our setting, the probability that the highest valuation equals $x$ depends on the meeting technology and is given by $1 - \phi(\lambda(1 - F(x)), \lambda)$, i.e., the probability that there are no buyers with valuations above $x$.

The intuition for the last claim is the following. Suppose there is a gap $(x_1, x_2)$ in the support of the queue at a seller in this submarket, i.e., no buyers with values between $x_1$ and $x_2$ attempt to meet the seller. If a buyer with value $x \in (x_1, x_2)$ then chooses to visit this seller, his payoff will be a weighted average of $U(x_1)$ and $U(x_2)$, as follows from equation (17). Since the market utility function is convex, this weighted average will lie above the market utility function. This leads to a contradiction.

One may have expected that allowing for general meeting technologies would severely complicate the payoff functions in (competing) auction theory. We have shown here that our alternative representation of the meeting technology $\phi$ avoids such complications. In particular, agents' expected payoffs retain the same structure but simply depend on transformations of $\phi$ instead of transformations of $F$.

**Example.** To better understand the above results, consider a bilateral meeting technology with $P_0(\lambda)$ strictly convex. Suppose that the measures of sellers and buyers are both equal to 1. Almost every buyer has value $a$, i.e., buyers with values other than $a$ have measure 0. As in proposition 4, we do not consider optimality of seller behavior and take the posted mechanisms as given; in particular, suppose half of the sellers posts a second price auction with reserve price 0 (market $A$), while the other sellers post a second price auction with reserve price or entry fee $r$, satisfying $0 < r < a$ (market $B$).[22]

In order to solve for buyers' optimal strategy, suppose that market tightness in markets $A$ and $B$ are equal to $\lambda^A$ and $\lambda^B$, respectively. Except in the corner solution in which all buyers with value $a$ visit market $A$, buyers with value $a$ must then be indifferent between visiting market $A$ and $B$. That is,

$$Q_1(\lambda^A)a = Q_1(\lambda^B)(a - r)$$

subject to the buyer availability constraint that $\lambda^A + \lambda^B = 2$.[23] The above equation implies $Q_1(\lambda^A) < Q_1(\lambda^B)$. Therefore, using the notation from proposition 4, we have $p(a, s_A) = Q_1(\lambda^A) < Q_1(\lambda^B) = p(a, s_B)$, where $s_A$ is an arbitrary seller in market $A$ and $s_B$ is an arbitrary seller in market $B$.

Next, we consider buyers with values other than $a$, even though they have measure 0. If their value $x$ satisfies $r < x < a$, then visiting market $A$ will result in a payoff of $Q_1(\lambda^A)x$ and visiting market $B$ will result in a payoff of $Q_1(\lambda^B)(x - r)$. Since $x < a$ and $Q_1(\lambda^B)r = a(Q_1(\lambda^B) - Q_1(\lambda^A))$, it follows that

$$Q_1(\lambda^A)x > Q_1(\lambda^B)(x - r).$$

Hence, buyers with valuation $x$ will visit market $A$ only. Therefore, $U(x) = Q_1(\lambda^A)x$ for $x < a$, and $p(x) = Q_1(\lambda^A)$. If $x > a$, a similar logic applies, implying

$$Q_1(\lambda^A)x < Q_1(\lambda^B)(x - r),$$

such that buyers with valuation $x$ will visit market $B$ only. Therefore, $U(x) = Q_1(\lambda^B)(x - r)$ for $x > a$ and $p(x) = Q_1(\lambda^B)$.

In sum,

$$p(x) = \begin{cases} Q_1(\lambda^A) \text{ if } x < a \\ Q_1(\lambda^A) \text{ if } x = a \text{ and } x \text{ visits market } A \\ Q_1(\lambda^B) \text{ if } x = a \text{ and } x \text{ visits market } B \\ Q_1(\lambda^B) \text{ if } x > a \end{cases}$$

[22] For this specific example, a reserve price and an entry fee are equivalent. This is not true in general.

[23] Note that $Q_1(\lambda) = \phi_\mu(\lambda, \lambda)$ by equation (7).

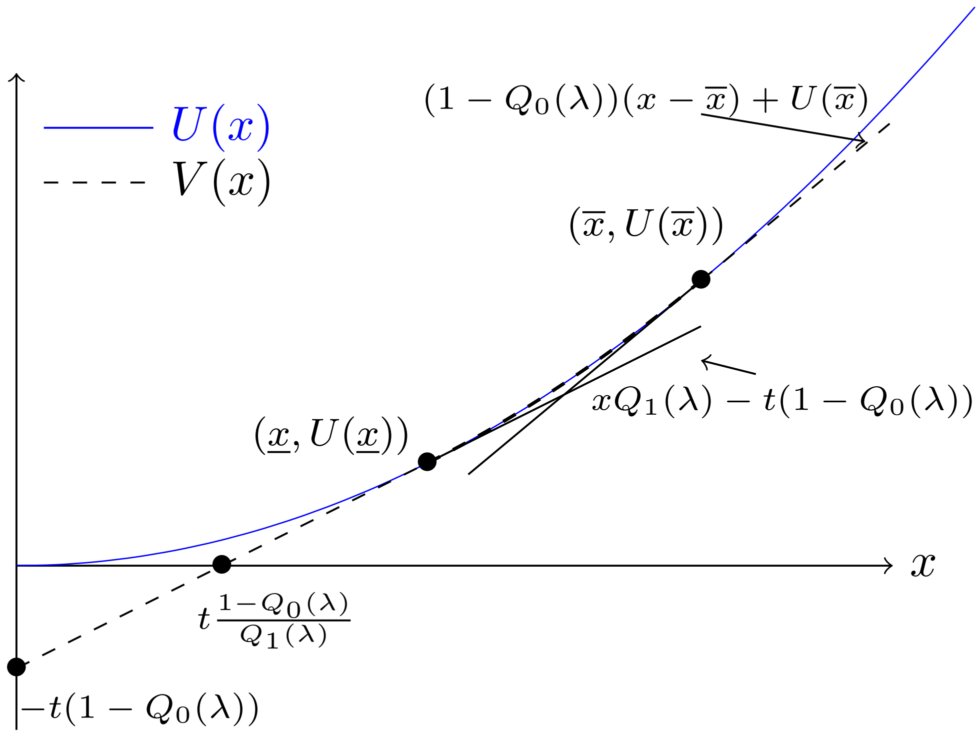


Figure 2: Supporting line

and

$$U(x) = \begin{cases} Q_1(\lambda^A)x \text{ if } x \leq a \\ Q_1(\lambda^B)(x - r) \text{ if } x \geq a \end{cases}$$

Hence, for any $x$, we have $U(x) = U(0) + \int_0^x p(z)dz$, where it does not matter whether we set $p(a)$ equal to $Q_1(\lambda^A)$ or $Q_1(\lambda^B)$. $U(x)$ is differentiable everywhere except at $a$.

**Payoffs Under Auctions and Fees.** Lemma 2 established payoffs under general efficient mechanisms. If we focus on a submarket in which sellers post auctions with entry fees, then a more specific expression for buyers' payoffs can be derived, as we establish in the following lemma.

**Lemma 3.** *Consider a seller posting an auction with entry fee $t$, who attracts a queue $\lambda$ of buyers with valuations in $[\underline{x}, \overline{x}]$. The expected payoff $V(x)$ of a buyer with value $x$ from visiting this submarket is*

$$V(x) = \begin{cases} xQ_1(\lambda) - t(1 - Q_0(\lambda)) & \text{if} \quad x < \underline{x}, & \text{(19a)} \\ U(x) & \text{if} \quad \underline{x} \leq x \leq \overline{x}, & \text{(19b)} \\ (1 - Q_0(\lambda))(x - \overline{x}) + U(\overline{x}) & \text{if} \quad \overline{x} < x \ , & \text{(19c)} \end{cases}$$

*where* (19a) *and* (19c) *are the supporting lines of the (convex) market utility function* $U(x)$ *at the points* $(\underline{x}, U(\underline{x}))$ *and* $(\overline{x}, U(\overline{x}))$*, respectively.*

*Proof.* See appendix A.7. □

Lemma 3 shows that there is a close connection between $\lambda$ and $\underline{x}$ and $\overline{x}$ through the supporting lines of the convex function $U(x)$. This observation is almost trivial but instrumental for understanding the relation between the entry fee and the queue in the next subsection.

Figure 2 illustrates lemma 3. By assumption, buyers with values below $\underline{x}$ or above $\overline{x}$ will not choose to visit this submarket. However, if they were to visit the submarket, their payoff would be given by $V(x)$, which is displayed by the dashed line. For values between $\underline{x}$ and $\overline{x}$, $V(x)$ coincides with $U(x)$. A buyer with value $x > \underline{x}$ will always trade as long as he successfully meets a seller, which happens with probability $1 - Q_0(\lambda) = \phi_\mu(0, \lambda)$. Hence, his payoff is given by the linear function $(1 - Q_0(\lambda))(x - \overline{x}) + U(\overline{x})$. In contrast, a buyer with value $x < \underline{x}$ will only trade if no other buyers meet the same seller, yielding a payoff equal to the linear function $xQ_1(\lambda) - t(1 - Q_0(\lambda))$.

## 5.2 Efficiency

In a decentralized market, in order to maximize his expected profit, a seller must choose a mechanism to attract a queue and the queue must be compatible with the market utility function. Below, we show that even if sellers can buy queues directly from a hypothetical market for queues (where the prices are given by the market utility function), they cannot do better than in the decentralized environment. In other words, the following two problems are equivalent.

1. *Sellers' Relaxed Problem.* There exists a hypothetical competitive market for queues, where the price for each buyer in the queue is given by the market utility function. Sellers choose a queue length $\lambda$ and a queue composition $F(x)$ to maximize

$$\pi = \int_0^1 \phi(\lambda(1 - F(x)), \lambda)dx - \lambda \int_0^1 U(x)dF(x), \tag{20}$$

   where the first term is total surplus (8) and the second term is the price of the queue.

2. *Sellers' Constrained Problem.* We have already described the seller's (constrained) problem in detail in Section 2. Contrary to sellers' relaxed problem, sellers must post mechanisms to attract queues of buyers. For any mechanism, the corresponding queue must be compatible with the market utility function, which means that it needs to satisfy equation (5). In this case, a seller's profit is again given by equation (20), but now queue length and queue composition depend on the posted mechanism.

Using compatibility as defined in equation (5), we have the following result.

**Proposition 5.** *Given any convex market utility function, any solution* $(\lambda, F)$ *to the sellers' relaxed problem is also compatible with an auction with an entry fee in the sellers' constrained problem, where the fee is given by*

$$t = -\frac{\int_0^1 \phi_\lambda(\lambda(1 - F(z)), \lambda)dz}{1 - Q_0(\lambda)}.$$

*Proof.* See appendix A.8. □

The intuition behind Proposition 5 is the following. In the sellers' relaxed problem, a seller will "buy" buyers with valuation $x$ until their marginal contribution $T(x)$ to surplus is equal to their marginal cost $U(x)$. Hence, if sellers can post a mechanism which delivers buyers their marginal contribution to surplus, then buyers' payoffs are equal to their market utility and the queue is compatible with the mechanism and the market utility function, as defined by equation (5). Proposition 5 argues that auctions with an entry fee can achieve this. To understand why this is the case, note that a buyer's marginal contribution consists of two parts: (i) a direct effect, representing the fact that the buyer may increase the maximum valuation among the group of buyers meeting the seller, and (ii) an indirect effect, representing the externalities that the buyer may impose by making it easier or harder for the seller to meet other buyers. As is well-known, auctions (without reserve prices or fees) provide buyers with a payoff equal to their direct contribution.[24] Buyers' indirect effect on surplus can then be priced by the entry fee. The combination of both instruments then guarantees that buyers' payoff is equal to $T(x)$, which yields the desired result.

There is one remaining issue about proposition 5: for a given auction with entry fee, there might be multiple queues compatible with the market utility function. Hence, even if a solution to the sellers' relaxed problem is compatible with an auction with entry fee, it is not clear that sellers will expect that solution to be the realized queue. Most of the literature resolves this issue by assuming that a deviating seller (expects that he) can coordinate buyers in such a way that the solution to sellers' relaxed problem becomes the realized queue.[25] As an alternative, we will introduce three weak additional restrictions on the meeting technology that guarantee uniqueness in the sellers' constrained problem, and which help us derive some additional results. For the moment, we assume that the uniqueness is always satisfied. By proposition 5, a seller's relaxed and constrained problem are equivalent in terms

[24]This is easiest to see in a second-price auction. Suppose that the highest and the second highest value are $x_2$ and $x_1$. Then, the payoff for the highest value buyer is $x_2 - x_1$, which is also his contribution to surplus. Other bidders receive zero and their contributions to the surplus of the auction are also zero. Extension of this result to other auction formats follows from revenue equivalence.

[25]See, for example, Eeckhout and Kircher (2010a,b).

of achieving the same outcome. That is, the directed search equilibrium is equivalent to a competitive market equilibrium for queues, which also coincides with the socially efficient planner's allocation.

**Proposition 6.** *The directed search equilibrium is efficient.*

*Proof.* See appendix A.9. □

Hence, we have shown that despite the potential presence of spillovers in the meeting process, business stealing externalities and agency costs, the competing mechanisms problem reduces to one where sellers can buy queues in a competitive market.

## 5.3 Uniqueness of Beliefs and Characterization

We now turn to a more detailed characterization of the queues of sellers posting auctions with an entry fee. In particular, we derive a number of conditions on the meeting technology such that the queue lengths vary monotonically with the fee and such that the queue composition is unique for a given queue length.

In Proposition 4, we established that buyer optimality implies that the market utility function $U(x)$ is always convex, irrespective of what mechanisms sellers post. When sellers post auctions with entry fee, one can derive sufficient conditions such that $U(x)$ is *strictly* convex on the open set $\{x \,|\, U(x) > 0\}$. One of these conditions is the following assumption, which will also play a role in subsequent results.

**Assumption 1.** *$Q_1(\lambda)$ is strictly decreasing in $\lambda$.*

This assumption states that in submarkets with longer queues, it is less likely that a buyer turns out to be the only one present in an auction. It is not restrictive in the sense that it is satisfied by all examples of meeting technologies that were listed above. Using this assumption, the following proposition then establishes sufficient conditions for strict convexity of $U(x)$:

**Proposition 7.** *Assume that the support of $G(x)$, the market-wide distribution of buyer types, is $[0, 1]$. Under assumption 1, $U(x)$ is strictly convex in equilibrium on the open set $\{x \,|\, U(x) > 0\}$.*

*Proof.* See appendix A.10. □

Following Proposition 7, we will assume that the market $U(x)$ is strictly convex on the open set $\{x \,|\, U(x) > 0\}$ in the remainder of this section. This allows us to show that the queue length varies monotonically with entry fee. To do so, we compare the queues of two arbitrary sellers, indexed by $i \in \{a, b\}$, who post an auction with an entry fee $t_i$, and attract a queue $\lambda^i$ of buyers, in which the lowest buyer type is $\underline{x}_i$. The following lemma establishes the relation between $\underline{x}_i$ and $\lambda^i$.

**Lemma 4.** *There is a unique $\underline{x}$ for a given queue length $\lambda$. Furthermore, under assumption 1, $\lambda^a > \lambda^b$ implies that $\underline{x}_b \geq \underline{x}_a$.*

*Proof.* See appendix A.11. □

The intuition behind lemma 4 can be easily seen from Figure 3. Since the market utility function is convex, the slope of a supporting line at $x_2$ is larger than that at $x_1$ if $x_2 > x_1$.

Similarly, a relation between a seller's queue length $\lambda^i$ and the *highest* buyer type $\overline{x}_i$ that he attracts can be established if we are willing to make the following assumption.

**Assumption 2.** $1 - Q_0(\lambda)$ *is (weakly) decreasing in* $\lambda$.

This assumption says that buyers are (weakly) less likely to meet a seller if the queue length in the submarket increases, which could be interpreted as a form of congestion. Like assumption 1, it is satisfied by all examples of meeting technologies that were listed above. Under this assumption, a lower $\overline{x}_i$ implies a longer queue, as the following proposition establishes.

**Lemma 5.** *There is a unique $\overline{x}$ for a given queue length $\lambda$. Furthermore, under assumption 2, $\lambda^a > \lambda^b$ implies that $\overline{x}_a \leq \overline{x}_b$. Furthermore, with non-rival meeting technologies, the queues attracted by sellers posting an auction with entry fee always have an upper bound $\overline{x} = 1$.*

*Proof.* See appendix A.12. □

This lemma, which does not require assumption 1, is the counterpart to lemma 4. As for that lemma, the intuition behind the result can be seen from Figure 2. Since the slope of a supporting line at $\overline{x}$ is $1 - Q_0(\lambda)$, if $1 - Q_0(\lambda)$ is decreasing, a longer queue implies a flatter supporting line at $\overline{x}$, hence the highest buyer value $\overline{x}$ is smaller.

Together with lemma 2, the second part of lemma 5 implies that under non-rival meeting technologies, each queue has a connected support with upper-bound 1 in equilibrium. Because we assumed non-rival meetings, if buyers with valuation 1 are absent, then a deviating buyer with valuation 1 will win for sure, assuming $Q_0(\lambda) = 0$. In other words, compared to other buyers in the queue, a buyer with valuation 1 will enjoy a large information rent if he decides to visit that seller, even higher than his market utility. In equilibrium this cannot happen because buyers with valuation 1 will adjust their visiting probability till the market utility constraint becomes binding again.

When the meeting technology is bilateral, the slopes of the supporting lines at $\underline{x}$ and $\overline{x}$ are the same, since $Q_1(\lambda) = 1 - Q_0(\lambda))$. As a result, $\underline{x}$ must be the same as $\overline{x}$. The above geometric argument therefore easily implies that complete market segmentation arises under bilateral meeting technologies. We discuss this result in more detail in Cai et al. (2016).

To prove uniqueness, we need a third assumption on the meeting technologies.

**Assumption 3.** $(1 - Q_0(\lambda))/Q_1(\lambda)$ *is (weakly) increasing in* $\lambda$.

If we rewrite $(1 - Q_0(\lambda))/Q_1(\lambda)$ as $1 + \sum_{k=2}^{\infty} Q_k(\lambda)/Q_1(\lambda)$, then the assumption states that with a higher buyer-seller ratio, it is relatively more likely that a buyer will meet competitors in an auction rather than being alone. Like assumption 1 and 2, assumption 3 is not restrictive; it is satisfied for each of our examples.

Making all of the above assumptions, we can then establish the following result.

**Lemma 6.** *Under assumptions 1, 2, and 3,* $t_a < t_b$ *if and only if* $\lambda^a > \lambda^b$.

*Proof.* See appendix A.13. □

The intuition behind lemma 6 readily follows from Figure 3. Consider two different queues $a$ and $b$. If queue $a$ is longer ($\lambda^a > \lambda^b$), then by lemma 4 the lowest type $\underline{x}_a$ in queue $a$ is smaller than the lowest type $\underline{x}_b$ in queue $b$. For bilateral meeting technologies, the intercepts between the supporting lines and the $x$-axis are $(t_a, 0)$ and $(t_b, 0)$, respectively. Since $\underline{x}_a < \underline{x}_b$, we can easily see from figure 3 that $t_a < t_b$. For non-rival meeting technologies, assuming $Q_0(\lambda) = 0$, the intercepts between the supporting lines and the $y$-axis are $(0, -t_a)$ and $(0, -t_b)$, respectively. Since $\underline{x}_a < \underline{x}_b$, we can again easily see from figure 3 that $t_a < t_b$. A similar logic holds for other meeting technologies.

Hence, for any strictly convex market utility function, sellers can adjust entry fees to attract queues of the desired length. The following proposition summarizes and establishes that the composition of these queues is uniquely determined.

**Proposition 8.** *Under assumption 1, 2, and 3, for a seller posting an auction with entry fee* $t$*, there is a unique queue* $(\lambda, F)$ *compatible with the market utility function* $U(x)$*. The support of* $F$ *is an interval* $[\underline{x}, \overline{x}]$*. If* $t_a < t_b$*, then* $\lambda^a > \lambda^b$*,* $\underline{x}_a \leq \underline{x}_b$*, and* $\overline{x}_a \leq \overline{x}_b$*. For non-rival meeting technologies,* $\overline{x}$ *is always* 1.

*Proof.* See appendix A.14. □

Since by proposition 5, the solution to the sellers' relaxed problem is compatible with the market utility function and an auction with entry fee, lemma 6 implies that a seller can (and will) always choose an appropriate entry fee such that a solution to the sellers' relaxed problem is the only queue compatible with the auction and the market utility function.

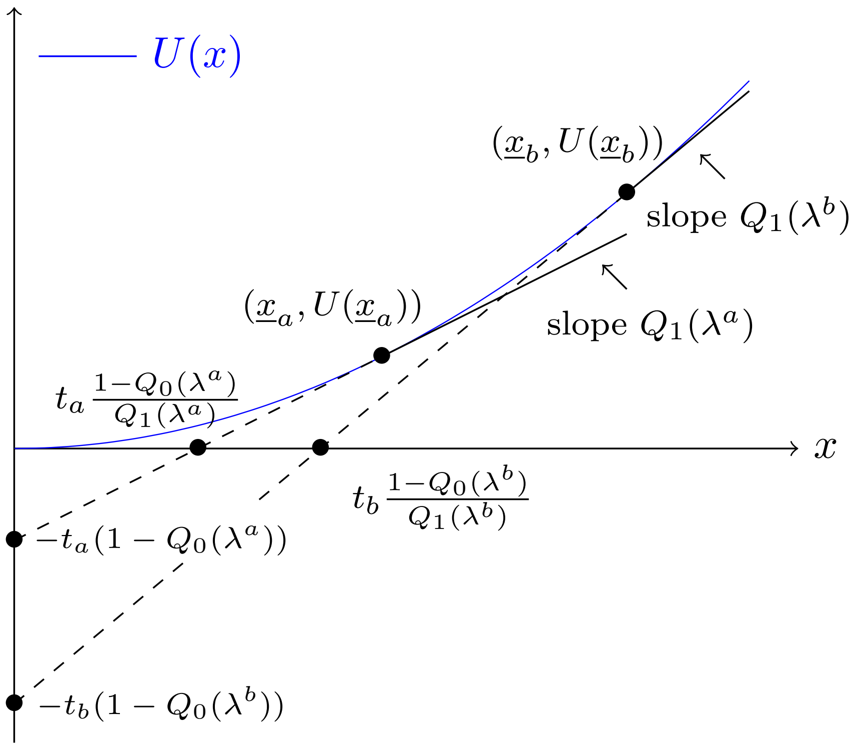


Figure 3: Relation between entry fee and queue length

Therefore, the solutions to a seller's relaxed and constrained problem coincide. That is, the directed search equilibrium is equivalent to a competitive market equilibrium for queues, which also coincides with the socially efficient planner's allocation.

While proposition 6 establishes efficiency of the directed search equilibrium and proposition 8 provides some characterization of the relation between entry fee, queue length, the lowest buyer type, and the highest buyer type, they do not characterize queue compositions. Under the following assumption, we can compare queue compositions between any two queues.

**Assumption 4.** $\phi_{\mu\lambda}(\mu, \lambda) \leq 0$ *for* $0 \leq \mu \leq \lambda$.

To understand this assumption, which is satisfied by e.g. bilateral and invariant meeting technologies, consider a queue with $\mu$ high-value buyers and $\lambda - \mu$ low-value buyers. By equation (7) and the subsequent discussion, $\phi_\mu(\mu, \lambda)$ is then the probability for a buyer to be part of a meeting in which all other buyers (if any) have low valuations, which is also the probability that a high-value buyer wins the auction with positive payoffs. Assumption 4 states that if we add more low-value buyers to the queue, then this probability will not increase. That is, low-value buyers create a weakly negative externality on high-value buyers. Note that this assumption implies assumption 2, as $d/d\lambda \, (1 - Q_0(\lambda)) = \phi_{\mu\lambda}(0, \lambda)$

by equation (7).[26]

The following proposition then characterizes our main result regarding the queue composition.

**Proposition 9.** *Under assumption 1, 3, and 4, consider two queues* $(\lambda^a, F^a(x))$ *and* $(\lambda^b, F^b(x))$. *If* $\lambda^a > \lambda^b$ *and* $\underline{x}_b < \overline{x}_a$, *then for any* $x \in [\underline{x}_b, \overline{x}_a]$,

$$\lambda^b \left(1 - F^b(x)\right) \geq \lambda^a \left(1 - F^a(x)\right).$$

*If the meeting technology is invariant, then* $\lambda^a (1 - F^a(x)) = \lambda^b \left(1 - F^b(x)\right)$ *for* $x \in [\underline{x}_b, 1]$.

*Proof.* See appendix A.15. □

Note that by Lemma 4 and 5, $\lambda^a > \lambda^b$ implies that $\underline{x}_a < \underline{x}_b$ and $\overline{x}_a \leq \overline{x}_b$. Only buyers with types belonging to $[\underline{x}_b, \overline{x}_a]$ are active in the two queues. For bilateral meeting technologies, the above proposition becomes void, because we have $\underline{x}_a = \overline{x}_a < \underline{x}_b = \overline{x}_b$, where the strict inequality is due to the assumption that $\lambda^a > \lambda^b$. For all non-rival meeting technologies, $\overline{x}$ is always 1 by Lemma 5. Hence, the support of $F^a$ contains the support of $F^b$. This is similar to Proposition 3 in Shimer (2005).

Unfortunately, a complete characterization of the equilibrium queues is not feasible, but progress can be made for special cases. For example, building on the results in this paper, Cai et al. (2016) establish that the equilibrium is perfectly separating (i.e. a separate submarket for each active type of buyer) for all $G(x)$ if and only if the meeting technology is bilateral. In contrast, the equilibrium features pooling of all agents in a single submarket for all $G(x)$ if and only if the meeting technology exhibits joint concavity, i.e. $\phi(\mu, \lambda)$ is concave in $(\mu, \lambda)$. For meeting technologies that are neither bilateral nor jointly concave, the equilibrium number of submarkets will generally depend on $G(x)$.

# 6 Two-Sided Heterogeneity

In this section, we show that our conclusions on existence, uniqueness and efficiency carry over to an environment in which sellers are heterogeneous. That is, we allow sellers to have different valuations $y$ for the good, satisfying $0 \leq y \leq 1$, and these valuations are sellers' private information. As before, each seller will post a direct, anonymous mechanism, and

[26] Moreover, assumption 4 is related to assumption 1. To see this, note that $Q_1(\lambda) = \phi_\mu(\lambda, \lambda)$, which implies $Q_1'(\lambda) = \phi_{\mu\mu}(\lambda, \lambda) + \phi_{\mu\lambda}(\lambda, \lambda)$. Since $\phi(\mu, \lambda)$ is always concave in $\mu$, we have $\phi_{\mu\mu}(\lambda, \lambda) \leq 0$, and hence assumption 4 implies that $Q_1(\lambda)$ is *weakly* decreasing in $\lambda$, a weaker version of assumption 1. Furthermore, by equation (7), we have $\phi_{\mu\mu}(\lambda, \lambda) = -Q_2(\lambda)/\lambda$. Therefore, assumption 4 implies assumption 1 if either one of the following two conditions holds: i) $P_2(\lambda) > 0$, or ii) $\phi_{\mu\lambda}(\lambda, \lambda) < 0$.

we require sellers with the same valuation to use the same (possibly mixed) strategy. To simplify the analysis, we will impose the following restriction on the meeting technologies.

**Assumption 5.** *Buyers impose (weakly) negative meeting externalities on each other, i.e.,* $\phi_\lambda(\mu, \lambda) \leq 0$ *for* $0 \leq \mu \leq \lambda$.

For bilateral meeting and invariant technologies, assumption 5 is automatically satisfied. For other meeting technologies, it is implied by assumption 4. To see this, note that $\phi_\lambda(0, \lambda) = 0$ because $\phi(0, \lambda) = 0$. Hence, assumption 4 implies $\phi_\lambda(\mu, \lambda) \leq \phi_\lambda(0, \lambda) = 0$.

## 6.1 Equilibrium

Proposition 2, which established expressions for (marginal) surplus, can easily be extended to the case with two-sided heterogeneity. In particular, in a submarket in which sellers have value $y$ and attract a queue $(\lambda, F)$, social surplus is

$$S_0(y, \lambda, F) = \int_y^1 \phi(\lambda(1 - F(x)), \lambda)\, dx, \tag{21}$$

where, compared to equation (8), the integration starts from the seller's valuation $y$ instead of 0. Similarly, the marginal contribution to surplus of a buyer with valuation $x$ equals,

$$T(x, y, \lambda, F) = \int_y^1 \phi_\lambda(\lambda(1 - F(z)), \lambda)\, dz + \int_{\min\{x,y\}}^x \phi_\mu(\lambda(1 - F(z)), \lambda)\, dz. \tag{22}$$

Note that if $x < y$, then a buyer with value $x$ does not directly contribute to surplus, i.e., the second term on the right hand side of equation (22) is zero. In this case, the buyer's marginal contribution to surplus consists only of spillovers, i.e., the first term on the right hand side of equation (22).

Next, we move to the decentralized market. We will again show that the relaxed and the constrained problem (as defined in Section 5) are equivalent for a seller with value $y$. Furthermore, a seller with value $y$ can solve its constrained problem by posting a second-price auction with some entry fee and a reserve price $y$.

First, note that since proposition 4 requires no restrictions regarding the seller's valuation, it continues to hold when sellers are heterogeneous. Hence, optimality of buyers' choices again implies that the market utility function is always convex.

In the relaxed problem, a seller with value $y$ will never select buyers with value less than $y$ as part of the optimal queue, since, by equation 22, the direct contribution to surplus of these buyers is zero and their indirect contribution is negative.[27]

[27]This was true by assumption in Section 5, where all buyers had positive values and sellers' value was zero.

Next, consider sellers' constrained problem. For a seller $y$, if all buyers in the market have values above $y$, then proposition 5 automatically applies: the solution to the seller's relaxed problem is compatible with an auction with reserve price $y$ and an entry fee. Because of assumption 5, the entry fee is non-negative, which implies that even when there are buyers with values below $y$ in the market, these buyers will choose to not participate in this auction. Proposition 5 therefore continues to hold.

To complete the argument, we only need to show that, as in Section 5, there is a unique queue compatible with the chosen mechanism (auction) and the market utility function. This is, however, almost trivial since for sellers with value $y$ who post auctions with reserve price $y$ and a (non-negative) entry fee, lemmas 3 to 6 continue to hold. This implies that there is a monotonic relation between entry fee and queue length. Thus, uniqueness is guaranteed and we have the following proposition.[28]

**Proposition 10.** *Under assumptions 1, 2, 3, and 5, the relaxed and the constrained problem of sellers are equivalent for any convex market utility function. Therefore, the directed search equilibrium is efficient.*

*Proof.* See the discussion above. □

Finally, we derive a dual statement of proposition 4. It links the equilibrium payoff of sellers with their selling probability. To do so, denote by $M(y)$ the equilibrium set of mechanisms posted by sellers with value $y$, select an arbitrary mechanism $\omega(y) \in M(y)$, and denote by $q(\omega(y))$ the probability that a seller of type $y$ successfully sells the object by using this mechanism. If we further denote by $\pi(y)$ the equilibrium payoff of a seller (in excess of his own value), then we can prove the following result.

**Proposition 11.** *In equilibrium, $q(\omega(y))$ is non-increasing in $y$ and $\pi(y)$ is decreasing and convex, satisfying*

$$\pi(y) = \int_y^1 q(z)\, dz.$$

*If $\pi(y)$ is differentiable at point $y_0$, then $q(\omega_0)$ is the same for every $\omega_0 \in M(y_0)$, i.e., the probability for sellers of value $y$ to successfully sell their object is the same across all mechanisms that they post.*

*Proof.* See appendix A.16. □

[28] Because the auctions now have reserve price $y$ instead of 0, there will be slight changes in some expressions. For example, in lemma 3, $V(x) = Q_1(\lambda)(x-y)-t(1-Q_0(\lambda))$ if $x < \underline{x}$, and in Figures 2 and 3 and in the proof of lemma 6, the intersection point between the supporting line at $\underline{x}$ and the $x$-axis will be $y+t(1-Q_0(\lambda))/Q_1(\lambda)$ instead of $t(1-Q_0(\lambda))/Q_1(\lambda)$. These changes should be clear from the context.

**Example.** We illustrate proposition 11 with an example. Suppose there is a measure 1 of buyers, who have values uniformly distributed on $[0,1]$. There is also a measure 1 of sellers, who almost all have a value 0. Suppose the meeting technology is urn-ball. As Cai et al. (2016) show, this has two implications which simplify the analysis: i) all agents will pool into one market (because $\phi$ is jointly concave), and ii) the equilibrium meeting fee is zero (because $\phi_\lambda = 0$). By proposition 4, the market utility function for buyers is

$$U(x) = \int_0^x \phi_\mu(1-z,1)\,dz = \int_0^x e^{-(1-z)}\,dz = e^{-(1-x)} - e^{-1}. \tag{23}$$

Now, consider a seller with value $y$, whose optimal mechanism is an auction with reserve price $y$. Suppose he attracts a queue with length $\lambda(y)$ and composition $F_y(x)$. For any $x$ in the support of $F_y(x)$, $V(x)$ coincides with $U(x)$ around $x$. Substituting equation (17) and (23) then yields

$$\phi_\mu(1-x,1) = \phi_\mu(\lambda(y)(1-F_y(x)),\lambda(y)),$$

Since $\phi_\mu(\mu,\lambda) = e^{-\mu}$, this implies $\lambda(y)(1-F_y(x)) = 1-x$. Hence, $F_y$ is uniform on $[\underline{x},1]$ and the queue length equals $\lambda(y) = 1-\underline{x}$ for some $\underline{x}$. Buyers with value $\underline{x}$ must obtain the market utility $U(\underline{x})$, which means that $\underline{x}$ has to satisfy $(\underline{x}-y)Q_1(1-\underline{x}) = U(\underline{x})$, or equivalently, $y = \underline{x} - (1-e^{-\underline{x}})$.

The probability that a seller with value $y$ successfully sells the good is therefore

$$q(y) = 1 - P_0(1-\underline{x}) = 1 - e^{-(1-\underline{x})}.$$

Next, we consider the profit function $\pi(y)$. It equals

$$\pi(y) = \int_y^{\underline{x}} \left(1-e^{-(1-\underline{x})}\right)dz + \int_{\underline{x}}^1 \left(1-e^{-(1-z)}\right)dz - \int_{\underline{x}}^1 \left(e^{-(1-z)} - e^{-1}\right)dz,$$

where the first two integrals on the right-hand side add up to total surplus and the last integral is the sum of buyers' expected utilities. A straightforward calculation then yields

$$\pi'(y) = \frac{d\pi(y)}{d\underline{x}}\frac{d\underline{x}}{dy} = -(1-e^{-(1-\underline{x})}).$$

Hence, as established in proposition 11, $\pi'(y) = q(y)$.

## 6.2 Positive Assortative Matching

Next, we will relate the types of sellers to the queues that they will attract in equilibrium. To do so, we need the following regularity assumption on the meeting technology.

**Assumption 6.** $P_0(\lambda)$ *is decreasing in* $\lambda$*.*

In equilibrium, sellers will post auctions with a reserve price equal to their valuations. Together with assumption 5, this implies that all buyers who visit a seller have values greater than the seller's value. In other words, if the queue length is $\lambda$, then the probability that the seller successfully sells the object is $1 - P_0(\lambda)$. By proposition 11, a seller with a higher value always has a lower selling probability. Therefore, for two sellers $a$ and $b$ who have values $y_a$ and $y_b$, satisfying $y_a < y_b$, it must be that $1 - P_0(\lambda^a) \geq 1 - P_0(\lambda^b)$, which implies $\lambda^a \geq \lambda^b$ because of assumption 6.

**Proposition 12.** *(Positive Assortative Matching.) Under assumptions 1, 2, 3, 5, and 6, in equilibrium, for any two sellers* $a$ *and* $b$ *with values* $y_a$ *and* $y_b$ *where* $y_a < y_b$*, the following must hold:* $\lambda^a \geq \lambda^b$*,* $\underline{x}_a \leq \underline{x}_b$*, and* $\overline{x}_a \leq \overline{x}_b$*. We call this the positive assortative matching case.*

*Proof.* See the discussion above. □

The intuition for this is the following. For a non-rival meeting technology like urn-ball, Albrecht et al. (2014) show that the buyers with the highest valuation randomize with equal probability over all sellers while buyers with low valuations do not visit sellers above some threshold.[29] High-type buyers do not care much about how many low type buyers there are because they will outbid them anyway in the auction. This gives rise to PAM in expectation. If buyers cause congestion on each other, high-type buyers prefer to visit segments with low $\lambda$'s and high valuation sellers to avoid congestion by low type buyers. If in addition we impose assumption 4, then Proposition 9 will continue to hold. We will avoid the repetition here to save space.

# 7 Conclusion

In this paper, we introduced a new function $\phi$ which makes the analysis of general meeting technologies tractable. Using this function, we show that in a large economy, despite the presence of private information and possible search externalities, the directed search equilibrium is equivalent to a competitive equilibrium where the commodities are buyer types and the prices are the market utilities. A seller can attract a desired queue by posting an auction

[29] They use the same logic as McAfee (1993).

with entry fee or subsidy. Furthermore, we introduced conditions on the meeting technology such that for any given market utility function, the queue attracted by an auction with fee is unique. This is necessary to establish the equivalence between the two equilibria.

# Appendix A Proofs

## A.1 Proof of Proposition 1

For a given sequence $P_n(\lambda)$, equation (6) defines the function $\phi$ immediately. For the reverse relationship, let $m(z,\lambda) \equiv \sum_{n=0}^{\infty} P_n(\lambda) z^n = 1-\phi(\lambda(1-z),\lambda)$ be the probability-generating function of $P_n(\lambda)$. Given $\phi$, the probability functions $P_n(\lambda)$, $n = 0, 1, 2, \ldots$, are then uniquely determined by

$$P_n(\lambda) = \frac{1}{n!}\frac{\partial^n}{\partial z^n} m(z,\lambda)\bigg|_{z=0} = \frac{(-\lambda)^n}{n!}\frac{\partial^n}{\partial \mu^n}(1-\phi(\mu,\lambda))\bigg|_{\mu=\lambda}.$$

□

## A.2 Proof of Proposition 2

When a seller meets $n \geq 1$ buyers, the surplus $x$ from the meeting is distributed according to $F^n(x)$. Hence, the expected surplus per seller in the submarket is

$$S(\lambda, F) = \sum_{n=1}^{\infty} P_n(\lambda) \int_0^1 x\, dF^n(x) = \int_0^1 \left(1 - \sum_{n=0}^{\infty} P_n(\lambda) F^n(x)\right) dx,$$

where we use the Dominated Convergence Theorem to interchange integration with summation. The rightmost integrand equals $\phi(\lambda(1-F(x)),\lambda)$, so the result follows.

Next, we calculate $T(x)$, the marginal contribution to surplus of a buyer with value $x$. First, we increase the measure of buyers with value $x$ by $\varepsilon$ and denote the new queue length and buyer value distribution as $\lambda'$ and $F'$ respectively. That is, $\lambda' = \lambda + \varepsilon$, while $\lambda'(1 - F'(z)) = \lambda(1 - F(z))$ for $z > x$ and $\lambda'(1 - F'(z)) = \lambda(1 - F(z)) + \varepsilon$ for $z \leq x$. Thus the average contribution to surplus by buyers with value $x$ is

$$\frac{S(\lambda', F') - S(\lambda, F)}{\varepsilon} = \frac{1}{\varepsilon}\int_0^x [\phi(\lambda(1-F(z))+\varepsilon, \lambda+\varepsilon) - \phi(\lambda(1-F(z)),\lambda)]\, dz$$
$$+\frac{1}{\varepsilon}\int_x^1 [\phi(\lambda(1-F(z)), \lambda+\varepsilon) - \phi(\lambda(1-F(z)),\lambda)]\, dz$$

Let $\varepsilon \to 0$, then the above equation converges to

$$T(x) = \int_0^1 \phi_\lambda \left(\lambda \left(1 - F\left(z\right)\right), \lambda\right)\, dz + \int_0^x \phi_\mu \left(\lambda \left(1 - F\left(z\right)\right), \lambda\right)\, dz. \tag{24}$$

Since total surplus is homogeneous of degree one in the measures of sellers and buyers of each type, the expression for $R$ follows from Euler's theorem, i.e., $R = S(\lambda, F) - \lambda \int_0^1 T(x)dF(x)$. To complete the proof, note that

$$\begin{aligned}\int_0^1 T(x)dF(x) &= \int_0^1 \phi_\lambda \left(\lambda \left(1 - F\left(x\right)\right), \lambda\right)\, dx + \int_0^1 \int_0^x \phi_\mu \left(\lambda \left(1 - F\left(z\right)\right), \lambda\right)\, dz\, dF(x) \\ &= \int_0^1 \phi_\lambda \left(\lambda \left(1 - F\left(x\right)\right), \lambda\right)\, dx + \int_0^1 \int_0^1 1_{z \leq x} \phi_\mu \left(\lambda \left(1 - F\left(z\right)\right), \lambda\right)\, dF(x)\, dz \\ &= \int_0^1 \phi_\lambda \left(\lambda \left(1 - F\left(x\right)\right), \lambda\right)\, dx + \int_0^1 \left(1 - F(z)\right) \phi_\mu \left(\lambda \left(1 - F\left(z\right)\right), \lambda\right)\, dz\end{aligned}$$

where in deriving the second equality above we used Fubini's theorem to change the order of integration. □

## A.3 Proof of Lemma 1

Assume $\phi_\lambda(\mu, \lambda) \geq 0$. Consider a submarket in which not all buyers have value zero, then $T(0) \geq 0$ by equation (9). Hence, buyers' marginal contribution to surplus is always non-negative in this case.

Assume $\phi_\lambda(\mu, \lambda) \leq 0$. Since $\phi(\mu, \lambda)$ is concave in $\mu$, $\phi(\mu, \lambda) - \mu\phi_\mu(\mu, \lambda) \geq 0$. Again consider a submarket in which not all buyers have value zero, then $\phi_\lambda(\mu, \lambda) \leq 0$ implies $R \geq 0$ in equation (10). That is, sellers' marginal contribution to surplus is always non-negative in this case. □

## A.4 Proof of Proposition 3

First, note that by equation (12), total surplus is a convex combination of the surpluses generated by individual submarkets. If the planner's solution requires that all buyers are active (i.e., no idle market for buyers), then the constraint of equation (14) is binding for all $j = 1, \ldots, n$. By equation (12), (13), and (14), the maximum social surplus as a function of the buyer endowment $(\lambda_1, \ldots, \lambda_n)$ is the concave hull of the function $S_0(\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i)$. The epigraph of $-S_0(\lambda_1^i, \lambda_2^i, \ldots, \lambda_n^i)$ is connected and lies in $\mathbb{R}^{n+1}$, hence by the Fenchel-Bunt Theorem (see Theorem 18 (ii) of Eggleston, 1958), which is an extension of Caratheodory's theorem, it suffices to create $k$ submarkets, where $k \leq n + 1$.[30]

If the optimum requires that some buyers are idle, then no sellers should be idle. A technical problem in this case is that in a submarket with only buyers, the queue length is infinite and surplus in (11) is not well defined. We can circumvent this problem by rewriting (11), and then using the same argument as above. To do so, define $\psi(a, b, c) = a\phi(b/a, c/a)$ if $a > 0$ and $\psi(0, b, c) = 0$. Surplus created in a submarket with a measure of $s$ sellers and a measure $b_i$ of buyers with value $x_i$ is then

$$S_0 = \sum_{j=1}^{n} (x_j - x_{j-1})\psi(s, B_j, B_1),$$

where $B_j = b_j + \cdots b_n$. Since there is no submarket with idle sellers, the total measure of buyers in a submarket, $B_1$, is always strictly positive. Hence, the above expression can be written as

$$S_0 = B_1 \sum_{j=1}^{n} (x_j - x_{j-1})\, \psi\left(\frac{s}{B_1}, \frac{B_j}{B_1}, 1\right).$$

Note that the arguments of function $\psi$ are finite and therefore well-defined for each submarket. We can then normalize the total measure of buyers to be 1. In this case, total surplus is a convex combination of the bivariate function $\psi(\cdot, \cdot, 1)$, and we can again use the above argument. □

## A.5 Proof of Proposition 4.

The strategy of a buyer with valuation $x$ is: (i) a probability distribution over the mechanisms (including inactivity) to visit and (ii) a value to report when the mechanism is not inactivity. Given the mechanisms posted by sellers, suppose that the set of sellers that a buyer with valuation $x$ visits is $S(x)$, and the probability that the buyer receives the object when visiting seller $i \in S(x)$ and reporting $x$ by $p(x, i)$, with a corresponding expected payment $t(x, i)$.

First, we select one index $s(x) \in S(x)$ for each $x$. Then, by the incentive compatibility constraint (ICC), for any $x, z$,

$$U(x) \geq xp(z, s(z)) - t(z, s(z)), \tag{25}$$

i.e., buyers with valuation $x$ are always better of following their equilibrium strategies than

[30] Here we use a special case of the Fenchel-Bunt Theorem: If a set $S \in \mathbb{R}^n$ is connected, then every point of the convex hull of $S$ can be represented by a convex combination of at most $n$ points in $S$.

mimicking any other type $z$. Therefore,

$$U(x) = \max_{z \in [0,1]} xp(z, s(z)) - t(z, s(z)).$$

Hence, $U(x)$ is the supreme of a collection of affine functions and must therefore be convex.

Furthermore, we can rewrite equation (25) in the following way.

$$\begin{aligned} U(z) = zp(z, s(z)) - t(z, s(z)) &\geq zp(x, s(x)) - t(x, s(x)) \\ &= U(x) + p(x, s(x))(z - x). \end{aligned}$$

So, $p(x, s(x))$ is the slope of a supporting line for the convex function $U(x)$. Therefore, $p(x, s(x))$ is a non-decreasing function. Since $U(x)$ is convex, it is absolutely continuous and differentiable almost everywhere. If $U(x)$ is differentiable at $x_0$, then

$$U'(x_0) = p(x_0, s(x_0)).$$

Since we have picked $s(x_0)$ out of $S(x_0)$ in an arbitrary way, this implies that for any $s_1, s_2 \in S(x_0)$, we have $p(x_0, s_1) = p(x_0, s_2) = U'(x_0)$. □

## A.6 Proof of Lemma 2

We use $V_n(x)$ to denote the expected payoff of a buyer with value $x$ when $n$ bidders are present in an auction. Taking the expectation with respect to $n$ and using $V_0(x) = 0$ yields

$$\begin{aligned} V(x) &= \sum_{n=1}^{\infty} Q_n(\lambda) V_n(x) = \sum_{n=1}^{\infty} Q_n(\lambda) \left( V_n(0) + \int_0^x F(z)^{n-1}\, dz \right) \\ &= V(0) + \int_0^x \left( \sum_{n=1}^{\infty} \frac{nP_n(\lambda)}{\lambda} F(z)^{n-1} \right) dz \\ &= V(0) + \int_0^x \left( \sum_{n=1}^{\infty} \frac{nP_n(\lambda)}{\lambda} F(z)^{n-1} \right) dz, \end{aligned}$$

where we have used equation (15) to substitute out $V_n(x)$. Therefore, using equation (7), we have

$$V(x) = V(0) + \int_0^x \phi_\mu(\lambda(1 - F(\widetilde{x})), \lambda) d\widetilde{x}.$$

The seller will receive $\pi_n$ in equation (16) with probability $P_n(\lambda)$. Therefore, for a given

$\lambda$, the expected profit of a seller is

$$\begin{aligned}\pi = \sum_{n=0}^{\infty} P_n(\lambda)\pi_n &= \int_0^1 \left(x - \frac{1-F(x)}{f(x)}\right) \sum_{n=0}^{\infty} P_n(\lambda) dF^n(x) \\ &= \int_0^1 \left(x - \frac{1-F(x)}{f(x)}\right) d\sum_{n=0}^{\infty} P_n(\lambda) F^n(x) \\ &= \int_0^1 \left(x - \frac{1-F(x)}{f(x)}\right) d\big(1-\phi(\lambda(1-F(x)),\lambda)\big),\end{aligned}$$

where we interchange integration and summation in the first line.

Finally, suppose that there is a gap $(x_1, x_2)$ in the set $\{x \,|\, V(x) = U(x)\}$, then buyers with values between $x_1$ and $x_2$ would earn an expected payoff strictly smaller than their market utilities and will not be present in the submarket, so $F(x) = F(x_1)$ for any $x \in (x_1, x_2)$. By equation (17), the payoff function $V(x)$ is linear between $x_1$ and $x_2$. Hence, for a buyer with value $x$ between $x_1$ and $x_2$, satisfying $x = \alpha x_1 + (1-\alpha)x_2$ for some $\alpha \in (0,1)$, it must be that $V(x) = \alpha V(x_1) + (1-\alpha)V(x_2) = \alpha U(x_1) + (1-\alpha)U(x_2) \geq U(\alpha x_1 + (1-\alpha)x_2) = U(x)$, where in the last inequality we used the fact that the market utility function is always convex (see proposition 4). We have thus reached a contradiction. □

## A.7 Proof of Lemma 3.

By equation (17), for $x < \underline{x}$, we have

$$V(\underline{x}) - V(x) = \int_x^{\underline{x}} \phi_\mu(\lambda(1-F(z)),\lambda)dz.$$

Since $x < \underline{x}$, we have $\phi_\mu(\lambda(1-F(z)),\lambda) = \phi_\mu(\lambda,\lambda) = Q_1(\lambda)$ by equation (7). Therefore, we obtain equation (19a).

Similarly, by equation (17), for $x > \overline{x}$, we have

$$V(x) - V(\overline{x}) = \int_{\overline{x}}^{x} \phi_\mu(\lambda(1-F(z)),\lambda)dz.$$

Since $x > \overline{x}$, we have $\phi_\mu(\lambda(1-F(z)),\lambda) = \phi_\mu(0,\lambda) = 1 - Q_0(\lambda)$ by equation (7). Therefore, we obtain equation (19c). Finally, by lemma 2 we have $V(x) = U(x)$ for $\underline{x} \leq x \leq \overline{x}$. □

## A.8 Proof of Proposition 5

In their relaxed problem, sellers select a queue $(\lambda, F)$ directly in a hypothetical competitive market. The expected payoff for a seller in this market is the difference between the surplus that he creates and the price of the queue. A buyer with valuation $x$ will therefore be in the support of $F$ only if his marginal contribution to this surplus equals the market utility $U(x)$, i.e., $U(x) = T(x)$, where $T(x)$ is given by equation (9) in proposition 2. If a buyer with valuation $x$ is not in the support of $F$, then $U(x) \geq T(x)$.

If we can find an entry fee $t$, such that $T(x) = V(x)$, then $(\lambda, F)$ is is also compatible with an auction with entry fee $t$ in the sellers' constrained problem. Let the entry fee $t$ be given by

$$t = -\frac{\int_0^1 \phi_\lambda(\lambda(1 - F(z)), \lambda)dz}{1 - Q_0(\lambda)}.$$

By equation (19a), we then have $V(0) = \int_0^1 \phi_\lambda(\lambda(1 - F(z)), \lambda)dz$. Furthermore, by equation (17), we have

$$V(x) = \int_0^x \phi_\mu(\lambda(1 - F(z)), \lambda)dz + \int_0^1 \phi_\lambda(\lambda(1 - F(z)), \lambda)dz = T(x).$$

Therefore, any queue chosen by an unrestricted seller who can buy queues directly at prices $U(x)$ is also compatible with an auction with entry fee. □

## A.9 Proof of Proposition 6

The sellers' relaxed problem boils down to a competitive market for buyer types. Therefore, the first welfare theorem implies and the equilibrium is efficient. Since the sellers' constrained problem is equivalent to the sellers' relaxed problem, the directed search equilibrium is also efficient. □

## A.10 Proof of Proposition 7

Suppose that $U(x)$ is not strictly convex on the open set $\{x \,|\, U(x) > 0\}$, then there exists an interval in which $U(x)$ is a straight line. Denote the interval by $(x_1, x_2)$. Since $U(x)$ is differentiable on this interval, the trading probability of buyers with value $x \in (x_1, x_2)$ is $U'(x)$ at all sellers that they visit by Proposition 4.

A queue at seller $i$ is characterized by $(\lambda, F)$, and the measure associated with $F$ is denoted by $\nu_F$. Consider all queues with $\nu_F(\{x \,|\, x_1 < x < x_2\}) > 0$, i.e., queues with positive measure on the interval $(x_1, x_2)$. Therefore, there exists a pair $x_1^*$ and $x_2^*$ such that $x_1 < x_1^* \leq x_2^* < x_2$, $x_1^*$ and $x_2^*$ belong to the support of $F$, and $\nu_F(\{x \,|\, x_1^* \leq x \leq x_2^*\}) > 0$.

If $P_0(\lambda) + P_1(\lambda) < 1$ or equivalently if $\phi(\mu, \lambda)$ is strictly convex in $\mu$ (see footnote 16), then for a buyer $x \in (x_2^*, x_2)$, the probability of winning if the buyer visits seller $i$, $p(x, i)$, satisfies

$p(x,i) > p(x_1^*) = U'(x_1^*) = U'(x)$. Therefore, by Proposition 4 the expected utility obtained by buyer $x$ from visiting seller $i$ is $U(x_1^*) + \int_{x_1^*}^{x} p(y,i)dy > U(x_1^*) + p(x_1^*)(x - x_1^*) = U(x)$. Hence, we have a contradiction.

Therefore, all buyers with $x \in (x_1, x_2)$ must visit sellers with queue $\lambda$ where $P_0(\lambda) + P_1(\lambda) = 1$, or equivalently $\phi(\mu, \lambda)$ is linear in $\mu$. Since $U(x)$ is differentiable for $x \in (x_1, x_2)$, by Proposition 4, $Q_1(\lambda) = U'(x)$, which implies that all such queues have the same length $\lambda$ (note that $Q_1(\lambda)$ is strictly decreasing by assumption 1). In the following, we denote it by $\lambda^*$. Furthermore, all such queues are associated with the same entry fee $t^*$, since $U(x) = Q_1(\lambda^*)(x - t^*)$. The expected profit of the sellers who post an entry fee $t^*$ is simply $P_1(\lambda^*)t^* = xP_1(\lambda^*) - \lambda^* U(x)$, where the latter must be independent of $x \in (x_1, x_2)$.

To simplify exposition, we assume that for $\lambda$ around $\lambda^*$, $P_0(\lambda) + P_1(\lambda) = 1$. Therefore, $x_1 P_1(\lambda) - \lambda U(x_1) = x_2 P_1(\lambda) - \lambda U(x_2)$. Their derivatives with respect to $\lambda$ can be written as $x_1 \lambda Q_1'(\lambda) + Q_1(\lambda)t$ and $x_2 \lambda Q_1'(\lambda) + Q_1(\lambda)t$, respectively. If $x_2 \lambda Q_1'(\lambda) + Q_1(\lambda)t < 0$, then a seller can increase his profit by charging a slightly higher entry fee than $t$ and attracting only buyers of type $x_2$ or slightly higher than $x_2$. The latter happens when $U(x)$ is differentiable at $x_2$. If $x_2 \lambda Q_1'(\lambda) + Q_1(\lambda)t \geq 0$, which implies $t > 0$ and $x_1 > 0$, then $x_1 \lambda Q_1'(\lambda) + Q_1(\lambda)t > 0$ because $Q_1'(\lambda) > 0$. In this case a seller can increase his profit by charging a slightly lower entry fee and attracting only buyers of type $x_1$ or a type slightly lower than $x_1$. The latter happens when $U(x)$ is differentiable at $x_1$. □

## A.11 Proof of Lemma 4

By lemma 3, $Q_1(\lambda^i)$ is the slope of a supporting line (subgradient) for the market utility function at point $(\underline{x}_i, U(\underline{x}_i))$ for $i \in \{a, b\}$. Because $U(x)$ is assumed to be strictly convex, the subgradient determines point $\underline{x}_i$ uniquely (see, for example, Theorem 24.1 of Rockafellar (1970)).

Furthermore, by assumption 1, $\lambda^a > \lambda^b$ implies $Q_1(\lambda_b) > Q_1(\lambda_a)$. Since $U(x)$ is assumed to be strictly convex, $Q_1(\lambda_b) > Q_1(\lambda_a)$ implies that $\underline{x}_b \geq \underline{x}_a$. □

## A.12 Proof of Lemma 5

By lemma 3, $1 - Q_0(\lambda^i)$ is the slope of a supporting line (subgradient) for the market utility function at point $(\overline{x}_i, U(\overline{x}_i))$ for $i \in \{a, b\}$. Similar to the proof of Lemma 4, the subgradient determines point $\overline{x}_i$ uniquely when $U(x)$ is strictly convex. Futhermore, by by assumption 2, $\lambda^a > \lambda^b$ implies $1 - Q_0(\lambda^a) \leq 1 - Q_0(\lambda^b)$, which implies $\overline{x}_a \leq \overline{x}_b$ by the strict convexity of $U(x)$.

For non-rival meeting technologies, $1 - Q_0(\lambda)$ is constant, which implies that the highest type of buyer must be the same across all submarkets. Since buyers with value 1 (the highest value) must visit all submarkets (otherwise no buyer will be active in the market), in all submarkets the highest buyer value is 1. □

## A.13 Proof of Lemma 6

We will prove the result in two steps. First, we show that a queue length $\lambda$ uniquely determines an entry fee $t$. Second, we show that $\lambda^a > \lambda^b$ if and only if $t_a < t_b$.

For the first part: as shown in Lemma 4, the queue length $\lambda$ uniquely determines the lowest buyer type $\underline{x}$. Furthermore, the supporting line associated with subgradient $Q_1(\lambda)$ can be written as $U(\underline{x}) + Q_1(\lambda)(x - \underline{x})$. By lemma 3, the supporting line is also given by $xQ_1(\lambda) - t(1 - Q_0(\lambda))$. Therefore, the entry fee $t$ is given by $-(U(\underline{x}) - \underline{x}Q_1(\lambda))/(1 - Q_0(\lambda))$, and therefore uniquely determined by $\lambda$.

For the second part, we will use the following geometric observation, which can be easily seen from figure 3.

**Lemma.** *Consider two supporting lines $a$ and $b$ at point $(\underline{x}_a, U(\underline{x}_a))$ and $(\underline{x}_b, U(\underline{x}_b))$ and with slopes $Q_1(\lambda^a)$ and $Q_1(\lambda^b)$, respectively. If $Q_1(\lambda^a) < Q_1(\lambda^b)$, then the intercept between the supporting line $a$ and the $x$-axis is strictly smaller than the intercept between the supporting line $b$ and the $x$-axis. A similar statement holds between the intercepts between the supporting line and the $y$-axis.*

*Proof.* First, consider the intercepts between the supporting lines and the $x$-axis. By the definition of the supporting line,

$$U(\underline{x}_a) > U(\underline{x}_b) + Q_1(\lambda^b)(\underline{x}_a - \underline{x}_b), \tag{26}$$

where the strict inequality is due to the strict convexity of $U(x)$. This implies that

$$\underline{x}_b - \underline{x}_a > \frac{U(\underline{x}_b)}{Q_1(\lambda^b)} - \frac{U(\underline{x}_a)}{Q_1(\lambda^b)} > \frac{U(\underline{x}_b)}{Q_1(\lambda^b)} - \frac{U(\underline{x}_a)}{Q_1(\lambda^a)},$$

where the second inequality follows from $Q_1(\lambda^a) < Q_1(\lambda^b)$. As the intercept between the supporting line and the $x$-axis is $\underline{x} - U(\underline{x})/Q_1(\lambda)$, the desired result follows.

Next, consider the intercepts between the supporting lines and the $y$-axis.

Equation (26) and $Q_1(\lambda^a) < Q_1(\lambda^b)$ also imply that

$$U(\underline{x}_b) - U(\underline{x}_a) < \underline{x}_b Q_1(\lambda^b) - \underline{x}_a Q_1(\lambda^b) < \underline{x}_b Q_1(\lambda^b) - \underline{x}_a Q_1(\lambda^a).$$

As the intercept between the supporting line and the $y$-axis is $U(\underline{x}) - \underline{x}Q_1(\lambda)$, the desired result follows. □

Since $Q_1(\lambda^a) < Q_1(\lambda^b)$ if and only if $\lambda^a > \lambda^b$ by assumption 1, the above geometric result implies that

$$\lambda^a > \lambda^b \Leftrightarrow t_a \frac{1 - Q_0(\lambda^a)}{Q_1(\lambda^a)} < t_b \frac{1 - Q_0(\lambda^b)}{Q_1(\lambda^b)} \Leftrightarrow t_a(1 - Q_0(\lambda^a)) > t_b(1 - Q_0(\lambda^b)),$$

where we have written the intercepts in terms of entry fees, using lemma 3. To prove the second part of the lemma, we now distinguish three different cases. First, if $t_a < 0 < t_b$, then the proof is immediate: the second and third inequality of the above equation hold, hence $\lambda^a > \lambda^b$.

Second, if $0 < t_a < t_b$, then we will prove by contradiction. Assume $\lambda^a \leq \lambda^b$, then we have $(1 - Q_0(\lambda^a))/Q_1(\lambda^a) \leq (1 - Q_0(\lambda^b))/Q_1(\lambda^b)$ by assumption 3. Multiplying this inequality with $t_a < t_b$ gives $t_a(1 - Q_0(\lambda^a))/Q_1(\lambda^a) < t_b(1 - Q_0(\lambda^b))/Q_1(\lambda^b)$, which implies $\lambda^a > \lambda^b$. Hence, we have reached a contradiction.

Finally, if $t_a < t_b < 0$, then again we will prove by contradiction. Assume $\lambda^a \leq \lambda^b$, then we have $1 - Q_0(\lambda^a) \geq 1 - Q_0(\lambda^b)$ by assumption 2. Multiplying this inequality with $t_a < t_b$ gives $t_a(1 - Q_0(\lambda^a)) > t_b(1 - Q_0(\lambda^b))$, which implies $\lambda^a > \lambda^b$. Hence, we have again reached a contradiction.

Hence, we have established hat $\lambda$ is a strictly increasing function of $t$, i.e. $t_a < t_b$ if and only if $\lambda^a > \lambda^b$. □

## A.14 Proof of Proposition 8

Most of the statements follow directly from lemma 2, 4, 5, and 6. The only thing left to prove is "for a given seller posting an auction with entry fee, if there are two queues of the same length that are compatible with the market utility function, then the queue compositions must be identical."

By lemma 4 and 5, a queue length $\lambda$ determines the lowest buyer value $\underline{x}$ and the highest buyer value $\overline{x}$ uniquely because of the strict convexity of $U(x)$. If $P_0(\lambda) + P_1(\lambda) = 1$, then $Q_0(\lambda) + Q_1(\lambda) = 1$ and $\underline{x} = \overline{x}$, as discussed in the main text. Hence, the result is trivially true. In contrast, if $P_0(\lambda) + P_1(\lambda) < 1$, then $Q_0(\lambda) + Q_1(\lambda) < 1$ and $\underline{x} < \overline{x}$. Consider $x \in (\underline{x}, \overline{x})$. By lemma 2, $U(x) = V(x) = U(\underline{x}) + \int_{\underline{x}}^{x} \phi_\mu(\lambda(1 - F(z)), \lambda)dz$. Therefore,

$\phi_\mu(\lambda(1-F(x),\lambda) = U'(x)$ almost everywhere, which determines $F(x)$ almost everywhere since $\phi(\mu,\lambda)$ is strictly concave in $\mu$. Furthermore, since $F$ is right-continuous, the above procedure determines $F$ uniquely. □

## A.15 Proof of Proposition 9

By Lemma 4 and 5, $\lambda^a > \lambda^b$ implies that $\underline{x}_a < \underline{x}_b$ and $\overline{x}_a \leq \overline{x}_b$. Hence, only buyers with types $x \in [\underline{x}_b, \overline{x}_a]$ are active in both queues. For $i \in \{a, b\}$, if $P_0(\lambda^i) + P_1(\lambda^i) = 1$, or equivalently $Q_0(\lambda^i) + Q_1(\lambda^i) = 1$, then $\underline{x}_i = \overline{x}_i$ by Lemma 4 and 5 (see also figure 2). In other words, the set $[\underline{x}_b, \overline{x}_a]$ becomes empty. We therefore only consider $P_0(\lambda^i) + P_1(\lambda^i) < 1$, which implies that $\phi(\mu, \lambda^i)$ is strictly concave and increasing in $\mu$ given $\lambda^i$.

By equation (17), for almost all $x \in [\underline{x}_b, \overline{x}_a]$, we have $\phi_\mu(\lambda^a(1-F^a(x)), \lambda^a) = U'(x) = \phi_\mu\left(\lambda^b\left(1-F^b(x)\right), \lambda^b\right)$. Since $F^a(x)$ or $F^b(x)$ are right continuous, for all $x \in [\underline{x}_b, \overline{x}_a)$, we have

$$\phi_\mu(\lambda^a(1-F^a(x)), \lambda^a) = \phi_\mu\left(\lambda^b\left(1-F^b(x)\right), \lambda^b\right). \tag{27}$$

We then prove the proposition by contradiction. Suppose that $\lambda^b\left(1-F^b(x)\right) < \lambda^a(1-F^a(x))$ for some $x \in [\underline{x}_b, \overline{x}_a)$. This implies

$$\phi_\mu(\lambda^a(1-F^a(x)), \lambda^a) < \phi_\mu\left(\lambda^b\left(1-F^b(x)\right), \lambda^a\right) \leq \phi_\mu\left(\lambda^b\left(1-F^b(x)\right), \lambda^b\right),$$

where the first inequality is because $\phi(\mu, \lambda^a)$ is strictly concave in $\mu$ and the second is because of assumption 4. The above inequality is at odds with equation (27). Hence, we have reached a contradiction.

In the special case of invariant meeting technologies, $\overline{x}^a = \overline{x}^b = 1$ by Lemma 5. Moreover, $\phi(\mu, \lambda)$ is strictly concave in $\mu$ and does not depend on $\lambda$. Hence, by equation (27), we have $\lambda^a(1-F^a(x)) = \lambda^b\left(1-F^b(x)\right)$. □

## A.16 Proof of Proposition 11

First, denote by $r(\omega)$ the expected revenue of a mechanism $\omega$. Then, the expected value of a seller $y$ (in excess of his own value) who posts $\omega$ is $r(\omega) - q(\omega)y$. Then

$$\pi(y) = \max_{\omega \in \mathcal{M}} \quad r(\omega) - q(\omega)y$$

where $\mathcal{M}$ is the set of all direct mechanisms including inactivity. Because $\pi(y)$ is the supreme of a collection of linear functions, it is convex. Furthermore, the optimality of $\omega(y)$ implies that

$$\begin{aligned}\pi(y) = r(\omega(y)) - q(\omega(y))y &\geq r(\omega(z)) - q(\omega(z))y \\ &= \pi(z) - q(\omega(z))(y - z).\end{aligned}$$

Therefore, $-q(w(z))$ is the slope of a supporting line at point $(z, \pi(z))$ for the convex function $\pi$. Note that $\pi(1) = 0$ because the highest buyer type is also 1. The rest of the proof is the same as in proposition 4. $\square$

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