



Government Debt Management: The Long and the Short of It

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Abstract

Our aim is to provide insights into some basic facts of US government debt management by introducing simple financial frictions in a Ramsey model of fiscal policy. We find that the share of short bonds in total U.S. debt is large, persistent, and highly correlated with total debt. A well known literature argues that optimal debt management should behave very differently: long term debt provides fiscal insurance, hence short bonds should not be issued and the position on short debt is volatile and negatively correlated with total debt. We show that this result hinges on the assumption that governments buy back the entire stock of previously issued long bonds each year, which is very far from observed debt management. We document how the U.S. Treasury rarely has repurchased bonds before 10 years after issuance. When we impose in the model that the government does not buy back old bonds the puzzle disappears and the optimal bond portfolio matches the facts mentioned above. The reason is that issuing only long term debt under no buyback would lead to a lumpiness in debt service payments, short bonds help offset this by smoothing out interest payments and tax rates. The same reasoning helps explain why governments issue coupon-paying bonds.

Solving dynamic stochastic models of optimal policy with a portfolio choice is computationally challenging. A separate contribution of this paper is to propose computational tools that enable this broad class of models to be solved. In particular we propose two significant extensions to the PEA class of computational methods which overcome problems due to the size of the model. These methods should be useful to many applications with portfolio problems and large state spaces.

JEL codes: C63, E43, E62, H63

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1 Introduction

Determining the optimal structure of government debt between long and short maturities, in other words *debt management*, is a much studied issue in fiscal policy. In almost all fiscal crises, including those since the Global Financial Crisis of 2007, the maturity structure of debt becomes a key variable. This suggests the necessity of jointly studying optimal fiscal policy and debt management. However, available models of optimal fiscal policy and debt management have difficulties in explaining basic observations.

A significant literature uses the framework of a Ramsey planner under full commitment, stochastic government expenditure and distortionary taxes. This literature reaches a strong presumption that governments should issue substantial quantities of long debt and that short debt should not be issued. In an endowment economy - where markets can be effectively completed by a bond portfolio with as many maturities as there are states of the world - Angeletos (2002) and Buera and Nicolini (2006) (ABN) show that optimal debt management requires the government to issue very large amounts of long debt and to purchase very large amounts of private short debt. This portfolio exploits variations in the yield curve so as to countervail shocks to the government budget constraint and achieve tax smoothing. Faraglia et al (2010) show that introducing capital accumulation and habit persistence causes the optimal issuance of long and short bonds to experience very dramatic reversals through time.

The clear predictions from these models conflict with observed debt management in an extreme way. In Section 2 we document empirically how for the US¹ *i*) short term debt constitutes a substantial proportion of total debt (between a quarter and a third) *ii*) governments issue only positive amounts of both short and long run debt and *iii*) the share of short debt has low volatility, it is highly persistent, and positively correlated with the share of long debt. These features are extremely different from the results in ABN. We see this as a challenge for DSGE models; it is very difficult to use these models to analyze debt management as long as they produce extreme recommendations that differ so dramatically from actual practice.

One obvious approach to fill this gap is the introduction of incomplete markets. Lustig et al. (2008) and Nosbusch (2008) go in this direction and they assume that the government is prevented from purchasing private bonds, in other words they impose fact *ii*) from the outset. They find that the presumption that governments should focus on issuing long term debt remains: optimal policy is to issue *only* long maturities, there is no role for short term debt. Therefore preventing lending to the private sector still contradicts all the facts that we mention as short bonds would not be issued at all.

In this paper we study thoroughly what type of financial frictions need to be introduced in order to obtain a behavior of debt management that is roughly like in the data. The above references all assume that each period the government buys back the entire stock of previously issued debt and the government simultaneously issues new bonds of varying maturities. In doing so the government restructures the debt each year and gross issuances are 100% of total debt. We refer to this assumption as "full buyback" (or sometimes simply as "buyback"). This assumption is popular in the literature for two reasons: first, it conveniently reduces the number of state variables in most applications;²

¹The evidence available suggests that these features are found in most OECD countries as well.

²Take a model with one short bond and one long bond that matures in N periods. Under full buyback only the

second, under *complete markets* optimal allocations, taxes and prices are the same with or without buyback.

But under incomplete markets the timing of payoffs influences the demand of an asset, therefore assuming buyback matters for the optimal supply of bonds. Furthermore, the assumption of full buyback is strikingly at odds with the data. We document carefully how the U.S. Treasury does not buy back much previously issued standard (non-callable) long bonds in the secondary market. Moreover, callable bonds are actually recalled shortly before maturity. As a consequence gross debt issuance each year is a relatively small and stable proportion of outstanding debt, the largest component of debt issued is short term.

Consistent with these observations we study optimal debt management and fiscal policy when the government is constrained not to buy back old long bonds. We find that introducing this constraint improves dramatically the fit of all the facts mentioned in the previous paragraphs.

The reason that assuming no buyback has such large effects on the bond portfolio is that the timing of payments of a long bond is very different whether or not it is bought back. If bonds of any maturity are bought back one year after issuance, as in all the references cited above, all bonds issued in period t pay something in period $t + 1$ and nothing after that, in that sense the timing of payments is the same for long and short bonds. Bonds only differ in the amount they pay, namely their price in the secondary market next period, so the correlation of bond prices with the deficit (fiscal insurance) is all that matters. However by ruling out buyback as we do (and as is done in practice) then the different cash flow properties of different maturities add another dimension to debt management.

In particular issuing only long bonds that are not repurchased has undesirable cash flow properties as it introduces lumpiness and large spikes at the time of redemption, leading to associated spikes in tax rates. By also issuing short term bonds the government can reduce this lumpiness of payments and therefore reduce the volatility of taxes. Long bonds are attractive because their return provides fiscal insurance, but they are less desirable because of the time profile of cash payments they produce, and the government has to trade off these two effects. This is why a mixed portfolio of short and long term debt is optimal. Further, we can explain why governments issue long bonds with positive coupon payments. Like short term bonds, coupon payments on long bonds help smooth tax volatility.

We find that introducing no buyback in standard models of Ramsey taxation improves dramatically the fit of many moments on debt management practice. We document carefully these moments in the data and compare them with the implications of several debt management models. This suite of models is designed to show the effects of various assumptions. The first model we consider is close to the above references, as bonds are bought back each period, but with incomplete markets as there are much fewer bonds than possible realizations of the shock. Then we introduce limits to lending, no buyback, coupons, and recalled bonds. In the last section we provide a measure of goodness of fit based on asymptotic confidence intervals, it shows that introducing no buyback improves the fit enormously, the model with coupons matches quantitatively most of the statistics considered.

We study the properties of the models combining analytic and numerical results. Solving numerically for incomplete market models with multiple bonds with maximum maturity of N is a

bonds issued last period are still in the economy, therefore bonds contribute only two variables to the state vector. By contrast, under no buyback all long bonds issued in the last N years are still in the economy, therefore the state vector must include at least $N + 1$ bonds. See section 3.1 for a detailed account of state variables in our model.

technically challenging task. Since portfolio constraints will be a crucial element in our models using linear approximations is not a good alternative. A further contribution of this paper is outlining computational methods based on the Parameterized Expectations Algorithm (hereafter PEA) of den Haan and Marcet (1990) to solve for optimal portfolios. In using this approach we confront two difficulties: i) the optimal portfolio choice is nearly indeterminate and ii) the size of the state space is very large. We solve the first by introducing the Forwarded States PEA and the second through using the Condensed PEA. Forwarded States works by approximating the integrand terms inside the expectations of the PEA with the date $t+1$ state variables rather than the usual date t . This enables us to circumvent an indeterminacy of the portfolio, and to solve a simple nonlinear system of first order conditions by integrating out future (unknown) values of the shock to government spending. The Condensed PEA is a technique which enables us to reduce the size of the state space, by forming an initial solution to the model, using a small size vector of core state variables, and subsequently testing the predictive power of the remaining state variables. These numerical procedures are of interest per se as they are likely to be useful in many other applications involving large portfolios and large state spaces.

Another contribution of the paper is to document carefully certain observations about the dynamics of bond portfolio and in particular the redemption and repurchase profile of the bulk of long term debt issued by the U.S. Treasury since the 20's. We find that 94% of standard non-callable US government long bonds have been redeemed at maturity, the remaining 6% can be accounted for by repurchases close to maturity and a well known unconventional episode in 2001. As for callable bonds they are often redeemed on their first call date which is typically close to maturity (5 years before maturity for 30-year bonds and 2 years before maturity for 10-year notes).

Alternative approaches to closing the gap between the model and the data have been to move away from the assumptions of no default and full commitment. An older literature (Calvo (1988) and Blanchard and Missale (1994)) consider moral hazard factors that lead governments to issue short term debt. Broner, Lorenzoni and Schmulker (2013), Aguiar and Amador (2013) and Arellano and Ramanarayanan (2012) explain the interaction between debt management and default. Debortoli, Nunes and Yared (2014) show that modifying the Angeletos (2002) model to allow for lack of full commitment leads to an increase in long term interest rates such that governments issue substantial amounts of short term debt as well as long term debt. Another strand of literature studies debt management in models where the demand for different maturities is determined by various features of the preferences of households. For example in Greenwood et al (2010) short term debt is valued by investors because it provides liquidity services to the private sector, long term debt valued by the government for tax smoothing purposes. Guibaud et al. (2013) propose a clientele based narrative suggesting that institutional investors, such as pension funds, have preferences for certain maturities (for example long term bonds)³.

Recall that our main point is that due to incomplete markets there is a role for short debt if long bonds are bought back at or close to maturity and that this brings the model much closer to observed debt management. This point is complementary to the stories described in the previous paragraph. The interplay between our main point and those stories should be of interest and it is

³These approaches add another layer to debt management which is not present in our model, namely that the volume and the maturity structure of government debt has a potentially important effect on the slope yield curve (see for example Greenwood and Vayanos (2010)).

yet to be explored.

In section 2 we describe and document carefully the facts we pursue to explain. Section 3 describes the basic model and some of its properties. Section 4 we introduce the numerical techniques of Forwarded states and Condensed PEA. Section 5 studies a model where all bonds are bought back each period. This is close to available models of debt management, but we study the case when the bonds available do not span the whole set of possible realizations, which is new to the literature. Section 6 introduces no buyback, we show this improves radically the fit of some key observations. Section 7 extends the results to bonds that pay coupons each year and to a model where bonds are recalled close to (but not at) maturity, the good fit of the model survives, it even improves with coupons, showing that the key issue is introducing no buyback. Section 8 summarizes some empirical observations and analyzes the fit of the models using a goodness of fit measure based on the asymptotic distribution. Section 9 concludes.

2 Some Stylized Facts on US Government Debt Management

The aim of this paper is to see what insights an incomplete market model of fiscal policy can provide into observed debt management practice in the United States. In this section we outline a series of key facts.⁴ Some of these facts are used to motivate modelling assumptions and others to examine the ability of different models to capture stylized facts. The figures reported are according to our own calculations using the CRSP data set, for a detailed description of the empirical analysis see Appendix A.1.

Figure 1 shows the share of the market value of short bonds issued as a proportion of the market value of the total stock of government debt for the United States over the period 1955-2011. We include as short term debt any outstanding (promised) payment by the government to bond holders which matures in less than one year.^{5,6} From Figure 1 we see the following :

Fact 1 Portfolio shares of long and short maturities are *both* substantial.

The share of short term debt is on average 36% and it ranges between approximately one quarter and one half. This share has a first order serial correlation of 0.91 and a standard deviation of 0.06.

⁴Some empirical evidence in favor of incomplete markets is the high persistence of debt and the positive co-movement of debt and primary deficit as emphasized in Marcet and Scott (2009). The setup in ABN would also fail to match those observations. Our debt management model does as well as the incomplete market models in Marcet and Scott (2009) in terms of explaining these observations. Since there is no improvement (but no worsening) in this aspect fitting the data we will not focus on these observations.

⁵The reported bond positions represent the governments total payout at maturity. We have stripped all coupons and added the sequence of payments as payouts of shorter maturity. Obviously excluded from this definition, as is standard in the literature, are implicit short term loans between the government and the private sector in terms of uncollected tax payments and during the month. A detailed description on the data and methodology is contained in the Appendix Section A1.

⁶Our focus is on government bonds and assets which are close substitutes to treasury securities. In broader terms, the US government as well as governments in other OECD economies may hold large stocks of financial assets including risky private loans, gold reserves and equity of state owned enterprises (as well as non-financial assets such as roads, bridges etc). As is standard in the literature we exclude these objects from our definition of net worth for the government.

Fact 2 The portfolio share is *never* zero or negative for short or long maturities.

Fact 3 Portfolio shares are highly stable over time, they exhibit low standard deviation and high serial (auto)correlation.

[Figure 1 About Here]

Fact 4 Short debt is positively correlated with long term debt.

We find that the correlation of the market value of short and long debt (both quantities measured over GDP) equals 0.79.⁷

These facts are in sharp contradiction to the usual recommendations from the Ramsey literature on debt management with its emphasis on issuing long term debt in large proportions and investing in short term assets. These models all maintain the assumption of “full buyback”,⁸ namely that every period the government buys back all outstanding bonds. We will show that these facts are still contradicted in an incomplete market model where the government has access to short and long bonds assuming full buyback in every period.⁹

Consider now Figure 2. It shows for the same sample period the total issuance of government debt (long and short maturities) over the total stock of debt held in every period.

[Figure 2 About Here]

From this we draw the following conclusion:

Fact 5 Total issuance of new debt each period is a fraction of the stock of outstanding debt.

Notice that Fact 5 tells us that the US government does not each period buy back the entire stock of debt and then reissue. If it did then the share displayed in Figure 2 would be equal to one.

In section 6.2 we study a model where we consider instead the other extreme assumption of “no buyback”, namely that the government does not repurchase any bond before redemption. We have been presented with the view¹⁰ that governments do repurchase old debt, so that full buy back can be seen as a simplification capturing this fact. It is true that actual debt management is somewhere in the middle of full- and no-buyback. However, the fact is that no-buyback is much closer to actual U.S. debt management. The remainder of this section documents this claim.

We now study empirically how much debt has been repurchased and when. Let us consider first the repurchases of standard non-callable bonds, which the government can only buy back through the secondary market. Although some empirical work has documented bond repurchases¹¹ we are not aware of an overall measure of their total magnitude for a long sample period. For this purpose

⁷We report the correlation of short debt over GDP in the text and in Table 3 to eliminate the trend in the data. This correlation continues to be very high regardless of the detrending method. For example, removing a linear-quadratic trend the correlation is 0.87.

⁸See the references we cite in the second and fourth paragraphs of the introduction.

⁹Lustig et al. (2008) and Nosbusch (2008) also study debt management in models of effectively incomplete markets.

¹⁰Mostly expressed to us in private conversations and seminar presentations.

¹¹Bloomenstein et al (2012) offer a survey of the debt management practice followed by OECD countries since 2001, they find that buybacks usually only occur close to maturity and aim to smooth redemption profiles and if the price of some maturity is thought to be out of line with the rest of the yield curve. Marchesi (2006) studies some repurchase episodes.

we have tracked all the non-callable bonds issued in the U.S. since the 1920s in the CRSP data set and recorded if and when non-callable long bonds were repurchased.

The total amount of redemptions by quarters is shown in Figure 3 and in percentages in Table 1. The horizontal axis of Figure 3 and the first column of Table 1 show the quarters prior to redemption when a given bond was repurchased, the bond is assigned to quarter zero if it was redeemed at maturity. According to Table 1 roughly 93.6% of all long maturity government debt was redeemed at maturity (0 quarters) and another 2% is bought back at some point up to one year before maturity. Since we calibrate our model to an annual frequency, this tells us that with regards to long debt issued since the 1920s the US has redeemed 96% of its long term non-callable debt either at the redemption date or repurchased the debt within a year of it. Hence we have

[Figure 3 and Table 1 About Here]

Fact 6 In their vast majority non-callable long bonds are redeemed at the maturity date and not before.

Callable bonds offer the government an option to redeem debt before maturity starting at a specified date without accessing the secondary market. For example a ten year bond can be redeemed eight years after its issuance, a thirty year bond twenty five years, etc. Since 1985 the U.S. Treasury has issued only non-callable bonds, but in some previous years (especially in the 1940s and the 1950s, see Table 2) the US government issued substantial amounts of callable bonds. We have tracked callable bonds in the CRSP data set and recorded if and when they were recalled, the details are in Appendix A.2.

Figure 14 shows the percentage of long bonds that are callable, it shows that until the early 80's these were a substantial fraction. Table 2 lists the issuance dates of callable debt in the US and the total amounts issued in such securities (1st column) and the fraction of the amount outstanding at every issuance which has been withdrawn from the market prior to the maturity date. In many cases the government has exercised the option to redeem the debt in full before the bond matures.

We find that the majority (around 80%) of callable debt is redeemed at the first call date, the remaining 20% is redeemed either 1 year before maturity or close to maturity. For most callable bonds issued the first call date is beyond 10 years after issuance, very few have a first callable date higher than 8 years.¹²

[Table 2 About Here]

This leads us to a further stylized fact:

Fact 7 Most callable bonds in the US have been redeemed at their first call date and after 10 years of their issuance date.

Facts 5-6-7 show that no buyback of 10-year bonds is actually very close to actual practice, while full buyback is very far from observed U.S. debt management. If anything callable bonds tend to be recalled at a fixed date close to maturity, a case we consider explicitly in section 7.2.

¹²For details see Table 5 and the discussion of "Call Windows For Callable Bonds in the US" in the Appendix, section A.2

In particular to show the robustness of our exercise to callable bonds, we will consider the case where the government removes long debt from the market at a fixed date prior to the maturity date. Since most callable bonds were bought at the first callable date this should be a reasonable approximation. We calibrate the fixed buyback date to be close to maturity, as the data tells us. Moreover, we do not try to offer microfoundations as to why governments do not buy back in this paper,¹³ instead we impose no buyback as a restriction in our main model and derive the consequences for debt management.

We will see that under no buyback there is a role for coupon payment, and in the robustness section we analyze a model that is consistent with the following

Fact 8 Long term bonds in the US pay constant coupons. These coupons are set so that at issuance long bonds trade very close to par.

The previous literature on debt management has focused on other empirical observations. Angeletos (2002) pointed out how the co-movement of the yield premium and government deficit in his model is roughly consistent with the data, this co-movement is precisely why issuing long bonds provides fiscal insurance.¹⁴ Marcet and Scott (2009) argue that the data shows high persistence of debt and a positive co-movement of debt and primary deficit, that this is incompatible with complete markets and, therefore, with the setup in ABN, but that incomplete markets fit these observations much better. Our debt management model matches these facts as well as the references mentioned in this paragraph so we do not mention them in the paper¹⁵.

3 The Model

Our benchmark model is of a Ramsey policy equilibrium with perfect commitment. It can be seen as introducing a long bond in Aiyagari et al. (2002) or as introducing further frictions in Angeletos (2002).

We assume a single representative household whose preferences over consumption, c_t , and leisure, x_t , are given by $E_0 \sum_{t=0}^{\infty} \beta^t (u(c_t) + v(x_t))$, where u and v are strictly increasing and strictly concave functions and $0 < \beta < 1$ is the discount factor,

The economy produces a single good that cannot be stored. The household is endowed with T units of time that it allocates between leisure and labour. Technology for every period t is given by:

$$(1) \quad c_t + g_t = (T - x_t)$$

¹³There are clear candidates to microfound the restriction of no buyback. One is to assume simply transaction costs. Another is the notion that tactical buybacks as an institutionalized feature of debt management have two (potential) disadvantages: First, they move the Treasury away from its practice of avoiding arbitrary changes to the outstanding amount of an issue and the maturity structure of debt (see Garbade and Rutherford (2007)). Second, they have potentially unpredictable outcomes on yields (see for example Greenwood and Vayanos (2010)) and hence open the possibility for costly operational mistakes and errors both by the Treasury and by market participants.

¹⁴Some recent empirical work by Brendt Lustig and Yeltekin (2012) points out that this mechanism is found in the data.

¹⁵A literature has focused on puzzles related to the behavior of the yield curve such as the failure of the expectations hypothesis, the average term premium and yield curve volatility. All previous work on debt management in equilibrium models that we are aware of, and indeed most of the DSGE literature, is inconsistent with these observations. Our paper is no exception. Ideally one would have a model of debt management consistent with these observations as well, we leave this for future research.

where g_t represents government expenditure assumed to be stochastic and exogenous and is the only source of uncertainty in the model. The representative firm maximizes profits. Both the household and the firm take prices and taxes as given.

The government engages in the following activities to finance spending: First, it levies distortionary taxes τ_t on labor income and second, it issues debt. Bond issuance of the government at period t is a vector $b_t = \{b_t^S, b_t^N\}$ where N denotes the long and S the short bond. Both are real, zero-coupon, riskless bonds: the short (long) bond promises to pay one unit of consumption in S (N) periods with certainty, we take the integer $S \geq 1$ to be much lower than the integer N .

Before we turn to a model with no buyback consistent with Facts 5-7 we start by assuming full buyback as in the existing literature of Ramsey debt management. This facilitates comparison since we depart from that literature in various dimensions.

Let p_t^i be the price of a bond of maturity $i = 1, \dots, N$ with $p_t^0 = 1$. The government budget constraint can be written as:

$$(2) \quad \sum_{i=\{S,N\}} b_t^i p_t^i = \sum_{i=\{S,N\}} b_{t-1}^i p_t^{i-1} + g_t - \tau_t(T - x_t)$$

The left side of this equation is the value of the bond portfolio issued this period. The first term on the right side is the market value of debt outstanding. The term $g_t - \tau_t(T - x_t)$ is the primary deficit. The household's budget constraint is given by:

$$(3) \quad \sum_{i=\{S,N\}} b_t^i p_t^i = \sum_{i=\{S,N\}} b_{t-1}^i p_t^{i-1} + (1 - \tau_t)(T - x_t) - c_t$$

We assume both the government and agents have full information, that is all variables dated t are contingent on observed histories of g^t .

3.1 Ramsey Problem

As is standard in the literature of Ramsey policy, we assume the government chooses tax and bond policies knowing the implied equilibrium quantities and seeking to maximize household utility. We first summarize the competitive equilibrium in a few equations.

From the consumer problem (maximization of utility subject to (3)) equilibrium bond prices satisfy $p_t^i = \beta^i E_t(\frac{u_{c,t+i}}{u_{c,t}})$, where $u_{c,t} \equiv u'(c_t)$. From the optimality condition for leisure we get $\tau_t = 1 - \frac{v_{x,t}}{u_{c,t}}$. Substituting these conditions and (1) in (2) we obtain the implementability conditions

$$(4) \quad \sum_{i \in \{S,N\}} b_t^i E_t(\beta^i u_{c,t+i}) = \sum_{i \in \{S,N\}} b_{t-1}^i E_t(\beta^{i-1} u_{c,t+i-1}) + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t).$$

As argued in Aiyagari et al. (2002) it is not possible to simplify further under incomplete markets, (4) has to be imposed for all t . In addition, following Aiyagari et al. (2002), we assume b_t^i has to

satisfy some ad hoc limits¹⁶

$$(5) \quad \frac{M_i}{\beta^i} \leq b_t^i \leq \frac{\bar{M}_i}{\beta^i}.$$

We will study cases where $\underline{M}_i < 0$ but also cases where $\underline{M}_i = 0$ (as in Lustig et al. (2008) and Nosbusch (2008)). We refer to the latter as a “No Lending” constraint. We scale both the upper and lower bounds of maturity i by the steady state price of debt for that maturity β^i so that the M 's are in units of the (steady state) market value of debt.

Using standard arguments we have that $\{c_t, b_t^S, b_t^N\}$ is an equilibrium sequence if and only if it satisfies (4) and (5) almost surely for all t . The Ramsey equilibrium solves a planner's problem choosing sequences $\{c_t, b_t^S, b_t^N\}$ to maximize the household's utility subject to (4) and (5) a.s. for all t . The Lagrangean for this planner's program is as follows:

$$(6) \quad \mathcal{L} = E_0 \sum_t \beta^t \left[u(c_t) + v(T - c_t - g_t) - \lambda_t \left(\sum_{i \in \{S, N\}} b_t^i \beta^i u_{c, t+i} - \sum_{i \in \{S, N\}} b_{t-1}^i \beta^{i-1} u_{c, t+i-1} \right. \right. \\ \left. \left. - g_t u_{c, t} + (u_{c, t} - v_{x, t})(g_t + c_t) \right) + \sum_{i \in \{S, N\}} \xi_{U, t}^i \left(\frac{\bar{M}_i}{\beta^i} - b_t^i \right) + \sum_{i \in \{S, N\}} \xi_{L, t}^i \left(b_t^i - \frac{M_i}{\beta^i} \right) \right].$$

Here $\xi_{L, t}^i$ and $\xi_{U, t}^i$ denote the multipliers on the lower and upper bounds respectively and λ_t is the multiplier of (4).

The first order conditions for the Ramsey optimum are:

$$(7) \quad u_{c, t} - v_{x, t} + \lambda_t [u_{cc, t} c_t + u_{c, t} + v_{xx, t}(c_t + g_t) - v_{x, t}] + \\ + u_{cc, t} \sum_{i \in \{S, N\}} (\lambda_{t-i} - \lambda_{t-i+1}) b_{t-i}^i = 0$$

$$(8) \quad \beta^i E_t (u_{c, t+i} \lambda_{t+1} - u_{c, t+i} \lambda_t) + \xi_{L, t}^i - \xi_{U, t}^i = 0 \quad \text{for } i = S, N.$$

Equation (7) represents the first order condition of consumption and (8) the analogous condition with respect to b_t^i . For the case of loose debt limits it holds that $\xi_{L, t}^i = \xi_{U, t}^i = 0$. Then, using the arguments of Aiyagari et al. (2002) we see that (8) states that the multiplier on the government budget λ_t evolves as a risk adjusted random walk with two risk adjusted measures, namely $u_{c, t+i}/E_t(u_{c, t+i})$ for $i = S, N$.

Extending the argument in Marcet and Marimon (2012) the optimal solution has a recursive formulation where the optimal tax schedule may be written as:

$$(9) \quad \tau_t = \tau(g_t, \lambda_{t-1}, \lambda_{t-2}, \dots, \lambda_{t-N}, b_{t-1}^S, \dots, b_{t-S}^S, b_{t-1}^N, \dots, b_{t-N}^N)$$

for a time-invariant function $\tau(\cdot)$ as long as we constrain $\lambda_{-1} = \dots = \lambda_{-N} = 0$. According to (9), and given the risk-adjusted martingale property of λ_t , changes in the value of the multiplier following a fiscal shock introduce a component to the optimal tax schedule that is very persistent, meaning that taxes will be permanently higher after the shock.

As noted by Angeletos (2002) and Buera and Nicolini (2006) (ABN), in the case that g_t is a

¹⁶As in Aiyagari et al. (2002) we assume for simplicity that these limits are tighter than the consumer's debt limits, thus the consumer is always at the margin.

Markov process taking only two possible values the optimal debt management strategy provides full insurance, λ_t is constant, the bond portfolio is time invariant and there is tax smoothing as under complete markets, in fact allocations, prices and taxes are the same as under complete markets. The intuition for this result is that in real business cycle models increases in the primary deficit are typically associated with drops in the ratio $E_t(\frac{u_{c,t+N-1}}{u_{c,t}})$, hence the value of long term debt drops when spending is high ensuring the intertemporal budget constraint holds without the need to change taxes considerably.

One first deviation from ABN is that we assume from the beginning that g_t has potentially a continuum of possible values and that the ad hoc constraints in (6) are tight enough so that the capital gains from holding long term debt are not sufficient to fully offset the effects of spending shocks on the government budget. Therefore, in our setup the complete market allocation is not reached, the government can not fully insure, the multiplier λ_t is not constant and the debt limit multipliers ξ are not necessarily equal to zero.

4 The Solution Method

In this section we describe the numerical procedure that is used to solve the model just stated and all the variations that follow below. We apply the widely used Parameterized Expectations Algorithm (PEA) of den Haan and Marcet (1990) to approximate a non-linear solution. As is well known, this approach involves approximating the model's conditional expectations with polynomials of the state space and solving the system of first order conditions to determine the paths of the endogenous variables.¹⁷

We need to introduce two modifications to PEA. The first modification is required because using the standard way of representing the first order conditions and the approximation yields a system of equations that is indeterminate. The second issue is that, as is clear from (9), the state space in our model may be huge. We therefore develop a methodology that reduces the dimensionality of the state variables.¹⁸ We refer to the first modification as Forward States PEA and to the second as the Condensed PEA.

To conserve space we mention here the principles of these methods. In the online Appendix B we describe the technical aspects of their implementation.¹⁹ We view the Forward States and the Condensed PEA as applicable to a broad class of models.

4.1 Forward States PEA

Let X_t be the vector of state variables that are relevant for the optimal allocation in period t

$$X_t = \{g_t, \lambda_{t-1}, \dots, \lambda_{t-N}, b_{t-1}^S, \dots, b_{t-N}^S, b_{t-1}^N, \dots, b_{t-N}^N\}.$$

¹⁷In the online Appendix B we explain the computational advantages of this method over some alternative approaches.

¹⁸Sometimes, in order to reduce the dimensionality of the state space, the literature assumes that bonds consist of geometrically decaying coupons where the decay rate can be varied to proxy for variations in maturity. Faraglia, Marcet, Oikonomou and Scott, A. (2014 (b)) show, in the context of models of optimal fiscal policy, that this approach is at best a weak approximation.

¹⁹In Faraglia, Marcet, Oikonomou and Scott (2014a) we provide a detailed description of how to solve many optimal fiscal policy problems with the PEA.

Solving the model requires finding sequences of allocations $\{c_t, b_t^S, b_t^N\}$ and multipliers $\{\lambda_t, \xi_{L,t}^i, \xi_{U,t}^i\}$, $i \in \{S, N\}$ solving the system of equations with conditional expectations that includes the first order condition for consumption (7), the optimality conditions for bonds (8), the government budget constraint (2). We write these conditions here

$$\begin{aligned}
(10) \quad & u_{c,t} - v_{x,t} + \lambda_t [u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t}] + u_{cc,t} \sum_{i \in \{S, N\}} (\lambda_{t-i} - \lambda_{t-i+1}) b_{t-i}^i = 0 \\
& E_t(u_{c,t+i} \lambda_{t+1}) = E_t(u_{c,t+i}) \lambda_t \quad \text{for } i = S, N \\
& \sum_{i \in \{S, N\}} E_t(\beta^i u_{c,t+i}) b_t^i = \sum_{i \in \{S, N\}} E_t(\beta^{i-1} u_{c,t+i-1}) b_{t-1}^i + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t).
\end{aligned}$$

In addition we need to impose the slackness conditions that arise from the debt limits.

4.1.1 The Conventional PEA Approach

We illustrate the indeterminacy problem mentioned above, we discuss in detail how we proceed in a region of the state space where the debt limits are non binding. Given the vector X_t and the conditional expectation terms, our aim is to solve the system of equations (10), to obtain the current value of consumption c_t , the bond quantities b_t^i , $i = S, N$, and the multiplier λ_t . For this purpose parameterizing expectations is to approximate the terms $E_t(u_{c,t+i})$ and $E_t(u_{c,t+i} \lambda_{t+i})$ as functions of the state vector X_t . Let us write:

$$(11) \quad E_t(u_{c,t+i}) = \Phi^i(X_t, \gamma^i) \quad \text{and} \quad E_t(\lambda_{t+1} u_{c,t+i}) = \Psi^i(X_t, \delta^i)$$

as the approximation of the conditional expectations under the PEA. Φ^i and Ψ^i belong to some class of polynomials that could approximate the conditional expectations arbitrarily well. The vectors γ^i and δ^i are the coefficients on these polynomials (say coefficients on state variables in X_t as well as their squares, cubes and so on, depending on the order of the approximating polynomial that we want to use).²⁰

System (10) has four equations that we hope will give a solution for the four variables $(c_t, b_t^S, b_t^N, \lambda_t)$ given the parameterized expectations. But this system is usually singular. Note that the two Euler equations imply

$$(12) \quad \lambda_t = \frac{\Psi^S(X_t, \delta^S)}{\Phi^S(X_t, \gamma^S)} \quad \text{and} \quad \lambda_t = \frac{\Psi^N(X_t, \delta^N)}{\Phi^N(X_t, \gamma^N)}.$$

Since the vector X_t contains only predetermined variables, (12) gives us two equations to solve for the variable λ_t so that (11) cannot be used to solve for λ_t .²¹

²⁰For the sake of clarity, we will represent the approximating functions using complete ordinary polynomials, though it should be noted that the technique may be applied to orthogonal polynomials (such as Chebyshev, Hermite and Legendre families). We utilize polynomials that are additively separable in the state variables as this allows us to estimate the coefficients with linear methods.

²¹Marcet and Singleton (1999) and den Haan (1995) already identified this problem in related models. They propose to approximate the term $E_t(u_{c,t+i} \lambda_{t+1} b_t^i)$ by a function $H^i(X_t, \delta^i)$, and the term $E_t(u_{c,t+i} \lambda_t)$ by a polynomial $G^i(X_t, \gamma^i)$. Then, theoretically, we could recover the bond holdings for different maturities using the following equation:

$$(13) \quad b_t^i = \frac{E_t(u_{c,t+i} \lambda_{t+1} b_t^i)}{E_t(u_{c,t+i} \lambda_t)} = \frac{H^i(X_t, \delta^i)}{G^i(X_t, \gamma^i)}$$

4.1.2 Solution through Forward States

Our proposal is to formulate conditional expectations as functions of current values of state variables. We accomplish this using the following two steps.

First, we forward one period each of the terms that we approximate, so the value of $\Phi^i(X_{t+1}, \gamma^i)$ is our numerical counterpart of the term $E_{t+1}(u_{c,t+i})$, and analogously the term $\Psi^i(X_{t+1}, \delta^i)$ is our approximation of $E_{t+1}(u_{c,t+i}\lambda_{t+1})$. Second, we invoke the law of iterated expectations to find the relevant expressions that enter in the first order conditions of the planner's program. We know for example that $E_t(u_{c,t+i}) = E_t E_{t+1}(u_{c,t+i})$ and thus our approximation of this term is given by $E_t \Phi^i(X_{t+1}, \gamma^i)$. Similarly, we use $E_t(\Psi^i(X_{t+1}, \delta^i))$ to approximate $E_t(u_{c,t+i}\lambda_{t+1})$. By substituting these expressions in the system of first order conditions we get:

$$(14) \quad \begin{aligned} & u_{c,t} - v_{x,t} + \lambda_t [u_{cc,t}c_t + u_{c,t} + v_{xx,t}(c_t + g_t) - v_{x,t}] + u_{cc,t} \sum_{i \in \{S,N\}} (\lambda_{t-i} - \lambda_{t-i+1}) b_{t-i}^i = 0 \\ & E_t(\Psi^i(X_{t+1}, \delta^i) - \Phi^i(X_{t+1}, \gamma^i)\lambda_t) \quad \text{for } i = S, N \\ & \sum_{i \in \{S,N\}} b_t^i \beta^i E_t(\Phi^i(X_{t+1}, \delta^i)) = \sum_{i \in \{S,N\}} b_{t-1}^i \beta^{i-1} \Phi^{i-1}(X_{t+1}, \delta^{i-1}) + g_t u_{c,t} - (u_{c,t} - v_{x,t})(g_t + c_t). \end{aligned}$$

Note that, differently from (12), in this modified system the integrands are a function of the variables that we wish to solve for. The vector X_{t+1} contains both current and lagged values of the multiplier and the bond quantities. The only variable that is unknown is next period's government spending shock, which may be integrated out, either analytically or with the use of simulations. In this way current values of b_t enter the two Euler equations and these can be used to obtain simulations for given $\Psi^i(\cdot, \delta^i)$ and $\Phi^i(\cdot, \gamma^i)$ and the first order conditions with respect to b_t^i hold. In the online Appendix we describe further the details of applying this procedure in the model at hand.

4.2 The Condensed PEA

As discussed previously, despite the simplicity of our model the state vector is very large. To give an idea of the number of variables in X_t , note that if we want to solve for a model where the government issues one-year and ten-year bonds (i.e. $S = 1$, $N = 10$), the number of state variables, including g_t , the lags of λ_t and the bond quantities, is 22. If we had allowed for all trades of maturities 1 to 10 years to be feasible, then the length of X_t would be 67, as every maturity m adds m lags of bond quantities to the state vector. Since debt limits will play a role in our model perturbation methods are not appropriate as they can not approximate well the solution both near and away from the debt limits, so we strive to approximate the non-linear solution. In this situation such a dimension of a state vector is difficult to handle.

With so many state variables our numerical methodology, may run into problems of close collinearity in the elements of X_t in the regressions run to find the parameters γ and δ . In PEA this leads the algorithm to either circle indefinitely or diverge. This, in our context, is all the more likely because of the near random walk property of the multiplier λ_t , which obviously extends the correlation to the lags of the bond quantities.

When we used this approach the algorithm diverges or circles indefinitely, it appears that asking the simple functional forms we use for H^i, G^i to approximate the object $u_{c,t+i}\lambda_{t+1}b_t^i$ is too much, while the approach of Forward States does work in practice.

But this multicollinearity is in a way encouraging: it means that in the optimal solution many of the elements of X_t influence the optimal solution only slightly. Therefore it is likely that a summary of the “relevant” information in X_t is sufficient for the solution and will be enough to obtain a good approximation. In some models it is known that this is strictly speaking the case. For example, under complete markets we know that all past bonds can be summarized in total wealth, so that one does not need to find a function of all past bonds in order to find the optimal solution in that case. In our case we may expect this to be approximately true.

We present in this section a method that condenses X_t . Intuitively, the method below seeks to reduce the dimensionality of X_t in two ways: first by keeping track only of elements of the state vector that are (nearly) perfectly correlated with the rest of the states, second some of the state variables are dropped because they may be (nearly) irrelevant for determining the optimal solution. This enables us to reduce the number of variables in the approximating polynomials and at the same time keep all the relevant state variables (or the information which is crucial for the solution) in the approximation.²²

Consider the approximation of the expectation

$$(15) \quad E_t(u_{c,t+i})$$

which in principle is a function of all elements X_t .²³ We partition the state vector into two parts: a subset of n state variables $\{X_t^{core}\} \subset \{X_t\}$, where n is small and an omitted subset of state variables $\{X_t^{out}\} = \{X_t\} - \{X_t^{core}\}$. Notice that in order to apply the strategy for Forward States described in Section 4.1 it must be at the minimum that (b_t^S, b_t^N) are included in $\{X_t^{core}\}$. Although in principle the approximating function $\Phi^i(\cdot, \gamma^i)$ may include higher order terms, for the sake of the exposition we illustrate our method assuming that $\Phi^i(\cdot, \gamma^i)$ is linear.

The idea is to first solve the model including only X_t^{core} as state variables and find a fixed point $\gamma^{i,f}$. We subsequently define the error:

$$(16) \quad \phi_{t+i} \equiv u_{c,t+i} - \Phi^i(X_t^{core}, \gamma^{i,f}).$$

If indeed this would be a good approximation then $E_t(u_{c,t+i}) \simeq \Phi^i(X_t^{core}, \gamma^{i,f})$ and the prediction error ϕ_{t+i} would be unpredictable with X_t^{out} . If this is the case for the solution found we claim the solution with core variables is the correct one. But if X_t^{out} can predict ϕ_{t+i} it must mean that some elements of X_t^{out} that are not perfectly correlated with X_t^{core} and matter for the optimality conditions. In order to try and find these elements of X_t^{out} we then find the linear combination of X_t^{out} that has the highest predictive power for ϕ_{t+i} , we add this linear combination (one more variable) to the set of state variables, solve the model again with X_t^{core} plus the new state variable, check if X_t^{out} can predict the new error ϕ_{t+i} and so on. Once we find that X_t^{out} that does not have any predictive power we will claim that we have found a sufficient summary of the whole state vector X_t .

Formally, given the set of core variables we define the condensed PEA as follows.

²²Reiter (2009) addresses a related issue in solving dynamic models with heterogeneous agents. He applies techniques used in systems and control theory to reduce the dimensionality of the features of the agents’ distribution of wealth.

²³Note that in practice all the conditional expectations in the model must be handled with the procedure we describe in this section. Moreover, it should be understood that when applied with the Forward States method outlined previously it should be that $i=S-1, N-1$ whereas the conventional approach sets $i=S, N$.

Step 1 Parameterize the expectation as

$$(17) \quad E_t(u_{c,t+i}) = (1, X_t^{core}) \cdot \bar{\gamma}^i.$$

Find values for $\bar{\gamma}^i \in R^{n+1}$, denoted $\bar{\gamma}^{i,f}$, that satisfy the usual PEA fixed point i.e. where the series generated by $(1, X_t^{core}) \cdot \bar{\gamma}^{i,f}$ causes this to be the best parameterized expectation of $u_{c,t+i}$.

This solution is of course based on a restricted set of state variables. It is therefore necessary to check if the omission of X_t^{out} affects the approximate solution. The next step orthogonalizes the information in X_t^{out} . This will be helpful to give good initial conditions for the next iteration and to arrive at a well conditioned fixed point problem in Step 4.

Step 2 Using a long simulation²⁴ run a regression of each element of X_t^{out} on the core variables. Let $X_{j,t}^{out}$ be the j -th element, we now run the regression

$$X_{j,t}^{out} = (1, X_t^{core}) \cdot \omega_j^1 + v_{j,t}^1$$

$\omega_j^1 \in R^{n+1}$ for $j = 1, 2, \dots, 2N + S + 1 - n$ and calculate the residuals

$$(18) \quad X_{j,t}^{res,1} = X_{j,t}^{out} - (1, X_t^{core}) \cdot \omega_j^1.$$

It is clear that $X^{res,1}$ adds the same information to X^{core} as X^{out} does, but $X^{res,1}$ has the advantage that it is orthogonal to X^{core} .

Step 3 Find $\alpha^1 \in R^{2N+S-n}$ through the following OLS regression:

$$(19) \quad \alpha^1 = \arg \min_{\alpha} \sum_{t=1}^T (u_{c,t+i} - X_t^{core} \cdot \bar{\gamma}^{i,f} - X_t^{res,1} \cdot \alpha)^2.$$

If α^1 is close to zero the solution with only X^{core} is sufficiently accurate and we can stop here. Otherwise there is evidence that more state variables should be added to the solution.

Step 4 Apply PEA adding $X_t^{res,1} \cdot \alpha^1$ as a state variable, i.e. parameterizing the conditional expectation as

$$E_t(u_{c,t+i}) = (X_t^{core}, X_t^{res,1} \alpha^1) \cdot \bar{\bar{\gamma}}^i$$

where $\bar{\bar{\gamma}}^i \in R^{n+2}$. Find a fixed point $\bar{\bar{\gamma}}^{i,f}$ for this parameterized expectation. Since $\bar{\bar{\gamma}}^{i,f}$ is a fixed point, since X_t^{core} and $X_t^{res,1}$ are orthogonal and since the linear combination α^1 has high predictive power, in order to find the fixed point $\bar{\bar{\gamma}}^{i,f}$ it makes sense to start the iterations at the initial conditions

$$\bar{\bar{\gamma}}_{(n+2) \times 1}^i = \begin{pmatrix} \bar{\gamma}^{i,f} \\ 1 \end{pmatrix}.$$

Go to Step 2 with $(X_t^{core}, \alpha^1 X_t^{res,1})$ in the role of X_t^{core} , check if a new linear combination is needed, etc.

²⁴This definition assumes we are interested in the steady state distribution, of course it could be modified in the usual way (i.e. running the model with many short samples) to take into account transitions. See, for example, Faraglia, Marcet, Oikonomou and Scott (2014a) for a detailed description.

A couple of remarks are in order. First, note that the Condensed PEA proposed in this section is designed to deal with a very large number of state variables. Our focus is on debt management and more broadly portfolio models but the method should be useful in many other applications with high-dimensional states including models with many sectors or heterogeneous agents.

Second, note that in the presence of many state variables the literature has often solved dynamic economic models by adding state variables one by one in some “order” until the next variable does not change much the solution found. For example, if many lags are needed the typical approach is to add the first lag, then the second lag, and so on. If at some step the solution changes very little it is claimed that the solution is sufficiently accurate. But it is easy to find reasons why this argument may fail. For instance, maybe the variables further down the list are more relevant, as is the case in our model since a simple inspection of the first equation in the system of FOC (10) suggests that the N -th lag of b^N and λ play a special role in the solution.²⁵ Or it can be that a linear combination of the remaining variables makes a difference but these variables do not make a difference one by one. The condensed PEA method gives a chance to all these variables to make a difference in the solution in only one step, so it will pick up a relevant variable regardless of the “order” of introduction (for example capturing the relevance of b_{t-N}^N, λ_{t-N}) and it will pick up the relevance of some combinations of state variables (for example capturing that under complete markets only total wealth matters).

5 Optimal Debt Management under Buyback

As explained in the introduction we are ultimately interested in a model where the government holds bonds until maturity. The literature has yet to explore the dynamics of debt management under the assumption of full buyback in a model with incomplete insurance.²⁶ To better understand the effects of each assumption we first consider in this section the standard model with full buyback, in the process we encounter features of this model that had not been previously discussed. Obviously, this model will fail Facts 5-8 by design, the issue is to study if it can address Facts 1-4.

The calibration of the model is as follows. We follow the literature and we take the long bond to mature in $N = 10$ years, while the short bond is a $S = 1$ -year bond. As discussed previously, the bounds \underline{M}_i and \overline{M}_i are in units of steady state market value of debt and we set the upper bounds \overline{M}_i equal to 150% of GDP. We first consider $\underline{M}_i = -\overline{M}_i$ and subsequently we impose $\underline{M}_i = 0$ to rule out lending from the government to the private sector. In words, the government in our model can issue debt for a total steady-state market value of debt three times as large as GDP if both bonds are at the upper bound. Since in our sample the US total government debt is less than GDP in all years we view these as a loose debt limit.²⁷

²⁵For another example, incomplete market models with a large number of agents need as state variable all the moments of the distribution of agents, which is an infinite number of state variables. Usually these models are solved first by using the first moment as a state variable, and checking that if the second moment is added nothing much changes. But it could be, of course, that the third or fourth moment are the relevant ones, especially since the distribution of wealth in the data is skewed.

²⁶The closest to this is Lustig et al. (2008). They study a model where the number of possible realizations equals the number of bonds but markets are effectively incomplete because of no-lending constraints. Our model is different in the extent of market incompleteness, since g has a continuum of realizations but only two bonds. Another difference is that we consider real bonds. Lustig et al. find that only the longest maturity is ever issued so their model can not speak to the observed dynamics of short bonds.

²⁷The government never hits the upper bound of total debt in our simulations (i.e. the overall market value is never as high as 300% of GDP). Debt constraints, however, have an impact on the optimal portfolio as they rule out large

To calibrate our model we follow Marcet and Scott (2009). We choose $\beta=0.95$, we set utility $u(c_t) + v(x_t) = \log(c_t) + \eta \frac{(x_t)^{1-\gamma}}{1-\gamma}$ and time endowment $T=100$. We choose a value of $\gamma = 2$ and target a value for η so that on average the household's leisure is 30% of the time endowment with taxes that balance the budget at the deterministic steady state, this gives $\eta = 12.857$.

Finally, our parameterization of the stochastic process of spending shocks is the following:

$$(20) \quad g_t = (1 - \rho_g)\bar{g} + \rho_g g_{t-1} + \epsilon_t.$$

We set $\rho_g=0.95$, $\sigma_\epsilon^2 = 1.44$. We further truncate the value of spending so that it always lies within an interval of 15% to 35% of steady state GDP. \bar{g} is chosen so that the ratio of spending to output is 25% in the deterministic steady state.

5.1 Buyback and Lending

We first consider the case where the government can both borrow and lend in the market. In Figure 4 we show a simulated path for the optimal portfolio plotting the market value of short and long bonds, while Figure 5 shows the path for the overall market value of debt for the same realization. This is meant to illustrate the government's deficit financing strategy in each period through the two maturities.

[Figures 4 and 5 About Here]

[Table 3 About Here]

In Table 3 we show the serial correlation, standard deviation and the mean of the share of short bonds over total debt, and the correlation of the market value of short debt with long debt normalized by output. The second row reports the empirical moments in the US whereas in the third row we show the computations for the variables in the version of the lending-buyback model that we now consider. The model-statistics reported are averages over one thousand samples generated by the model, each sample 60 years long. The initial value of government debt in each sample is 60% relative to GDP which is the corresponding starting value in our dataset.

The behavior of the model is as follows: as in other models of optimal policy under incomplete markets and no lower bound on government debt (e.g. Aiyagari et al (2002)) we find that in the long run the government accumulates savings. This is seen from Figure 5. The mechanism that leads governments to accumulate savings is well known so we do not motivate it and document it any further.

Also, we find that on average the value of the long bond issued $p_t^N b_t^N$ is much larger than the value of the short bond $p_t^S b_t^S$. The mean position in short term bonds is -43.27 while the mean position in long bonds is 40.31. This, combined with the fact that debt is on average negative, is why the model

stocks of long term debt and short term savings (Angeletos (2002) and Buera and Nicolini (2004)). In our simulations the overall debt level is rarely as high as 120-130% of GDP, under buyback and in the case lending is permitted, long term bonds may hit their upper bound constraint and short bonds their lower bound. We refer the reader to Faraglia, Marcet, Oikonomou and Scott (2014a) for further details on these model properties.

yields an average share of short term bonds which is positive and exceeds unity. In this average sense this model of incomplete markets echoes the complete market result of Angeletos, Buera and Nicolini (ABN) where the optimal portfolio was to issue long and save short.

However, in contrast to the complete market result, the optimal portfolio is not constant over time. There is an active management of the maturity structure, the share of long and short bonds changes through time. This is clear from Figure 4 and from the fact that the variance of the share of short bonds is very large.²⁸ We sometimes have very quick changes of these positions and in some periods we even have reversals, where $b_t^N < b_t^S$, as occurs in periods 120 to 150 of Figure 4. These reversals happen in 5% of the periods.

The reason for this high volatility of the positions is a combination of two model features. First, as is common in incomplete market models, *total* debt varies through time because it is used as a buffer stock to smooth out shocks. Second, we know that optimal short and long positions that achieve fiscal insurance under complete markets are very sensitive to changes in initial wealth, as is clear from standard computations in ABN. Combining the two facts it is not surprising that the positions on each bond change through time in our model, optimal policy under incomplete markets also seeks fiscal insurance and it uses debt as a buffer stock. In part the optimal policy here is to roughly mimic the effects of the positions in ABN but for a level of debt that changes from period to period.

The reversals that occur, for example, in periods 120 to 150 of Figure 4 are harder to understand given the results of ABN. We find, however, that the ABN prediction of issuing long and saving short in a complete markets model is not robust. In the online Appendix ²⁹ we show analytically a case where the predictions of ABN are reversed: for a standard utility function, N very large and sufficiently high initial savings of the government optimal issuance in a model of effectively complete markets implies $b_t^N < b_t^S$ and b_t^S very close to zero.

Proposition 1 Assume the innovations to government spending can only take two values $\epsilon_t \in \{\epsilon^H, \epsilon^L\}$ with some probabilities p^H and $(1 - p^H)$, utility $\log c_t - B \log l_t$ and initial government savings are sufficiently high. Consider increasingly long maturities N . Then, we have that

$$\begin{aligned} \beta^{N-1} b_t^N (g^t) &\rightarrow \bar{b} < 0 \\ b_{t-1}^1 (g^{t-1}) &\rightarrow 0 \end{aligned}$$

as $N \rightarrow \infty$.

This example, therefore, shows that even if markets are effectively complete you can get reversals and it justifies that reversals occur in our model.

5.1.1 Comparison of the ‘Lending’-Buyback Model with the Data

Are the model features consistent with the Facts reported in Section 2 of the paper? The answer is obviously not. In the model government debt is practically entirely long term, and therefore, the implied portfolio strategy fails badly in matching Fact 1.

²⁸In Angeletos, Buera and Nicolini the *value* of the shares of short bonds are not exactly constant but their variance is near zero since the position is constant and the price has very high serial correlation.

²⁹See section A.1 of the online Appendix.

Moreover, the standard deviation of the share of short term debt is huge and the first order autocorrelation is extremely low (0.22 in the model vs. 0.95 in the data). The correlation of short debt with long debt is negative, as could have been anticipated from a glance at Figure 4. Therefore the model observations are also inconsistent with Facts 1-4 reported in Section 2.

5.2 Buyback and No Lending

In order to give the model the best chance to match the data, we investigate in this section the properties of optimal debt management when the possibility of lending to the private sector is ruled out. We just assume (as in Lustig et al. (2008) and Nosbusch (2008)) an ad hoc lower bound on bonds $\underline{M}_i = 0$ for $i = S, N$. We leave the upper bounds \overline{M}_i and all other parameters as in the previous subsection.

We do not pursue to microfound this constraint, we just take the limit as given. Candidate stories for such microfoundations would be the uninsurable risk involved in holding private assets, and controversies over exactly what private assets should the government buy. In practice the U.S. Treasury does hold some private assets but they are usually small and they are held for reasons outside the model. In particular, the Treasury holds deposit accounts to face seasonal liquidity spikes, since we have a yearly model this is not an issue for us. The Fed and Treasury have purchased private assets during the financial crisis, but this has been done in order to face market disruptions that we also abstract from, in any case this has been a very rare event during our sample. While these are fascinating issues we leave them out of our study.

Figure 6 plots a sample path for this model. The solid line represents one year debt and the dashed line ten year debt. The simulation uses the same sequence of spending shocks as in Figure 4 in the previous section.

[Figure 6 About Here]

The result which emerges from the figure is that under ‘No Lending’ it is still optimal to issue mostly long term debt. Though there is some debt held in one year maturity, in many periods $b_t^1 = 0$. As in the previous section in which we allowed the government to both borrow and lend, long bonds have a hedging value to the government budget. The ‘No Lending’ model thus predicts that debt management is mostly driven by fiscal insurance considerations so that long maturity debt is still preferable to finance deficits. In periods where the overall debt level is high short bonds are issued, giving rise to a positive correlation between the share of short bonds and the overall market value of debt in the model in those periods. This is not surprising since for high enough total debt the short position in the model without limits can be positive.³⁰

5.2.1 Comparison of the ‘No Lending’-Buyback Model with the Data

In the third column of Table 3 we show the key moments related to Facts 1, 3, 4 for the model in this section. The average share of short term debt is roughly ten percentage points lower than in

³⁰Lustig et al. (2008) find that the no lending limit is always binding, the government issues only the longest maturity available to the market. They consider nominal debt, in that case inflation can be utilized for fiscal hedging purposes, long bonds enable the government to also smooth the distortions from inflation over time. When debt is real this additional benefit from long maturity is absent.

the data (26% in the model compared to 36% in the data), the standard deviation is three times larger than in the data (0.17 vs. 0.06). Therefore the volatility of the share of short bonds σ_{S_1}/\bar{S}_1 is four times larger in the model than in the data. The model also underpredicts the persistence of the share (0.79 vs. 0.91).

Related to this point, the share of bonds is equal to zero with positive probability while this never occurs in the data. In roughly 8% of all periods in our simulations, we encounter zeros in the value of the share so that Fact 2 is violated. Moreover, we find that in 14% of all periods the share is lower than 10 per cent and in 40% of all times it is less than 20 per cent. In the US data short term debt has never been below 26 per cent.

The model also fails to explain the correlation of short and long debt (Fact 4), this is positive in the model (0.12) but much lower than in the data (0.78).

By construction the model fails in matching Facts 5-8. We conclude that the ‘No Lending’-Buyback model is a fairly bad approximation of the historical observations on US debt management.

6 Optimal Debt Management under No Buyback

In the previous section we followed the literature in assuming that each period all outstanding government debt, independent of its maturity, is bought back and it is replaced with new bonds. Our discussion of Facts 5 to 7 in Section 2 showed that this is very far from actual practice.

We now study optimal debt when the government is constrained to never repurchasing old bonds (no buyback). As this analysis is largely unexplored in the theoretical literature we describe in detail the general setup and the Ramsey problem in this case. We maintain for now the assumption of zero coupons, and hence bonds offer a single payout at maturity. In the robustness section we study a model with coupons and the case where the government removes long debt from the market a few periods before it matures.

6.1 The Ramsey Program

We maintain the two bonds S, N as in the previous section. When government debt is held to maturity we can write the per period budget constraint as:

$$(21) \quad \sum_{i \in \{S, N\}} b_t^i p_t^i = \sum_{i \in \{S, N\}} b_{t-i}^i + g_t - \tau_t(T - x_t)$$

The left hand side of (21) corresponds to the market value of new debt issued in period t , the primary deficit is given by $g_t - \tau_t(T - x_t)$, and the value of the maturing debt is $\sum_{i \in \{S, N\}} b_{t-i}^i$.

The debt limits have to be modified in order to represent a constraint on the overall market value of debt comparable to the one in the previous section. Assume that we are in a steady state so that $p_t^i = \beta^i$ for maturity i . Then, given the issuances between t and $t - N + 1$ the market value of debt in N bonds still outstanding is: $\sum_{j=1}^N \beta^j b_{t-N+j}^N$. Therefore we normalize the debt constraints for

$i = \{S, N\}$ as

$$(22) \quad b_t^i \in \left[\frac{\underline{M}_i}{\sum_{j=1}^i \beta^j}, \frac{\overline{M}_i}{\sum_{j=1}^i \beta^j} \right]$$

Note that this puts the same limits as in the previous section for the value (in steady state) of debt in each bond, and for the total market value of debt.

Using similar substitutions as in section 3.1 the Lagrangian for the planner's program is

$$(23) \quad \mathcal{L} = E_0 \sum_t \beta^t \left[u(c_t) + v(T - c_t - g_t) - \lambda_t \left[\sum_{i \in \{S, N\}} b_t^i \beta^i u_{c, t+i} - \sum_{i \in \{S, N\}} b_{t-i}^i u_{c, t} \right. \right. \\ \left. \left. - g_t u_{c, t} + (u_{c, t} - v_{x, t})(g_t + c_t) \right] + \sum_{i \in \{S, N\}} \xi_{U, t}^i \left(\frac{\overline{M}_i}{\sum_{j=1}^i \beta^j} - b_t^i \right) + \sum_{i \in \{S, N\}} \xi_{L, t}^i \left(b_t^i - \frac{\underline{M}_i}{\sum_{j=1}^i \beta^j} \right) \right]$$

The first order conditions for the optimum are given by:

$$(24) \quad u_{c, t} - v_{x, t} + \lambda_t [u_{cc, t} c_t + u_{c, t} + v_{xx, t}(c_t + g_t) - v_{x, t}] + u_{cc, t} \sum_{i \in \{S, N\}} (\lambda_{t-i} - \lambda_t) b_{t-i}^i = 0$$

$$(25) \quad \beta^i E_t(u_{c, t+i} \lambda_{t+i} - u_{c, t+i} \lambda_t) + \xi_{L, t}^i - \xi_{U, t}^i = 0 \quad \text{for } i = S, N$$

where (24) and (25) are the first order conditions for consumption and for bonds respectively.³¹

6.2 Fiscal Hedging and Taxes Under No Buyback

Before we display the simulations we discuss two analytic results that help us understand the properties of the model. These results show that now there are two opposing forces, one in favor of issuing long bonds (fiscal insurance) and one in favor of issuing short bonds (smoothing cash payments).

6.2.1 In Favor of Long Bonds: Fiscal Insurance

First, we show that the notion that the government can better insure against spending shocks through issuing long term debt still applies. Notice that the outstanding bonds that do not mature at time t does not appear explicitly in the per period budget constraint (21), although these bonds do affect the government's future financing needs. To make this explicit we reformulate the budget constraint.

Let $s_t = \tau_t(T - x_t) - g_t$ be the primary surplus in t . From the flow budget constraint in $t + j$ we find

$$E_t(s_{t+j}) + E_t \sum_{i \in \{S, N\}} b_{t+j}^i p_{t+j}^i = E_t \sum_{i \in \{S, N\}} b_{t-i+j}^i$$

³¹The numerical procedure outlined in Section 4 of the paper is necessary to solve the equilibrium under no buyback. According to (25) each of the two Euler equations defines (uniquely) a value of λ_t under the conventional PEA, and the set of states for this model is still given by:

$$X_t = \{g_t, \lambda_{t-1}, \dots, \lambda_{t-N}, b_{t-1}^S, b_{t-S}^S, b_{t-1}^N, \dots, b_{t-N}^N\}$$

Therefore both the 'Forward States' methodology and the 'Condensed PEA' approach described in Section 4, can be fully applied in this context.

multiplying the above expression by p_t^j and summing over all $j = 0, 1, \dots, N-1$ we can write:

$$(26) \quad E_t p_t^N \sum_{i \in \{S, N\}} \sum_{j=0}^{i-1} p_{t+N}^j b_{t-i+j+N}^i + E_t \sum_{j=0}^{N-1} p_t^j s_{t+j} = E_t \sum_{i \in \{S, N\}} \sum_{j=0}^{i-1} p_t^j b_{t-i+j}^i$$

where we made use of the standard properties of bond prices that $p_t^i = E_t(p_{t+j}^{i-j} p_t^j)$.

Iterating forward, and taking $S = 1$, we can express the intertemporal budget of the government as

$$(27) \quad E_t \sum_{j=0}^{\infty} \beta^j \frac{u_{c,t+j}}{u_{c,t}} s_{t+j} = p^{N-1} b_{t-1}^N + p^{N-2} b_{t-2}^N + \dots + b_{t-N}^N + b_{t-1}^1$$

which gives the usual condition that the expected future discounted surpluses have to equal current total debt.

The intertemporal budget constraint in (27) shows a hedging effect of issuing long term debt analogous to the buyback model. Suppose there is a rise in g_t above the steady state level. This will cause the left side of this equation to go down. Since long bond prices go down when the primary deficit increases the right side of (27) will compensate this increase if b_{t-j}^N are positive. Therefore we may expect that issuing long bonds helps build a portfolio that absorbs shocks to primary deficit and it alleviates tax volatility.

To give some analytic basis to the intuitive argument in the previous paragraph we analyze in Section A.2 of the online Appendix the case when g can only take two values and no buyback. We find that, as in ABN, long term debt can be used to effectively complete markets by issuing long debt and saving in short debt. What differs now is that under no buyback, long bonds that were issued in the past but have not yet matured contribute to fiscal hedging and, in this case, the relative magnitude of the positions of long debt and short savings taken under complete markets, is smaller.

Given these observations we can expect that issuing long debt provides fiscal insurance also under no buyback in our model.

6.2.2 Against Long Bonds: Tax Volatility

It turns out that long bonds have a problem: they fabricate lumps in bond payouts that can lead to large volatility of taxes.

To make this point clear let us consider a special case where only long bonds are present and with a very simple structure for g . Assume:

- i)* no short bond can be issued or purchased, that is $\underline{M}_S = \overline{M}_S = 0$;
- ii)* g_t is not random, it is higher in the first period than in all the others: $g_0 > g_1 = g_2 = \dots$;
- iii)* debt limits on the long bond are not binding.

The first order conditions are still given by the budget constraints, (24) and (25), although the latter effectively does not hold for b_t^S due to assumption *i*). Equation (25) for b_t^N gives:

$$(28) \quad \lambda_t = \lambda_{t+N} \quad \text{for all } t = 0, 1, \dots$$

In other words, λ has an N - period cycle.

Furthermore, under the additional assumption

$$iv) b_{-i}^N = \bar{b} \text{ for } i = 1, \dots, N;$$

then $\lambda_1 = \lambda_2 = \dots = \lambda_{N-1}$.³²

Putting all this together implies that

$$\begin{aligned} \lambda_{tN} &= \lambda_0 \text{ for } t = 1, 2, \dots \\ \lambda_{tN+i} &= \bar{\lambda} \text{ for } i = 1, \dots, N-1, \text{ and } t = 0, 1, 2, \dots \end{aligned}$$

Equation (24) implies that this cycle also arises for consumption and taxes.

The evolution of taxes in this example is shown in Figure 7 assuming $b_{-i}^N = 0$ for $i = 1, \dots, N$.

The dashed line represents taxes in a three year maturity model ($N = 3$) and the crossed line in a ten year maturity ($N = 10$). As we can see from this Figure taxes are very volatile if only a long bond is issued and it is bought back when it matures. The higher tax needed in period $t = 0$ for the high g_0 reverberates every N periods even if there are no further high values of g , causing a large increase in taxes in periods Nt , while the high g_0 has no effect on taxes in the intervening periods. The solid (blue) line in Figure 7 shows the optimal allocation if only a short bond would be issued, it is clear that in this case the high g_0 affects taxes in all periods and the increase each period is much lower than the spikes in taxes for $N = 10$. This shows how issuing only a long zero-coupon bond induces tax volatility relative to issuing only a short bond.

Therefore, if we allowed the government to issue short bonds by setting $-\underline{M}_S = \overline{M}_S > 0$ it would be optimal to issue short debt in this economy, as this would achieve the complete market outcome.

To understand the reason for this result notice that through forward iteration on the budget constraints (21) we have

$$(29) \quad \sum_{j=0, N, 2N, \dots} \beta^j u_{c, t+j} (\tau_{t+j} (T - x_{t+j}) - g_{t+j}) = u_{c, t} b_{t-N}^N \quad \text{for } t = 0, 1, \dots, N-1.$$

This shows that if only one long bond is issued taxes can only be compensated every N periods. After the initial spending increase there is a rise in taxes and issuance of debt in $t = 0$. But the new debt issued at $t = 0$ has to be redeemed in N years at which point taxes have to increase for the accumulated interest and further debt has to be issued. It is pointless to increase taxes in intervening periods when spending is back at the steady state value. More precisely, it is not possible to reduce τ_0 by increasing, for example, τ_1 or τ_2 . The additional tax revenue at $t = 2$ cannot be utilized to reduce the debt accumulated at $t = 0$. The only effect of increasing τ_2 would be to reduce τ_{2+N} and therefore to fabricate even more volatility in the intervening periods. It is only possible to reduce τ_0 by increasing τ_N, τ_{2N} and so on.

[Figure 7 About Here]

The case displayed in Figure 7 is chosen to give the starkest result but the presence of these

³²To prove this notice that (28) and (24) imply $c_i = c_{i+Nt}$ for $i = 1, 2, \dots, N-1$ and all $t = 1, 2, \dots$. Together with (29) this implies $u_{c, i} (\tau_i (T - x_i) - \bar{g}) = u_{c, i} \bar{b} (1 - \beta^N)$, therefore $c_i = \bar{c}$ and $\lambda_i = \bar{\lambda}$, for $i = 1, 2, \dots, N-1$.

cycles is very robust.³³ The N -period cycle in λ is always present as implied by (25) for N . This cycle in λ will usually imply a pattern for taxes as in Figure 7. The result that issuing short bond is optimal in this example hinges on the assumption of having a lower bound on short debt \underline{M}_S that may allow for lending but that is sufficiently tight,³⁴ but strictly speaking it does not require a no-lending constraint $\underline{M}_S = 0$.

6.3 Optimal Debt Management under No Buyback

We now turn to the optimal portfolio under no buyback. Our simulations are displayed in Figures 8 and 9. All parameter values, including the M 's, are as in section 5. Figure 8 considers the case of the 'Lending' model, i.e. where the government is permitted to hold negative debt in both maturities. Figure 9 presents the simulation results under 'No Lending', when $\underline{M}_i = 0$ $i = S, N$.

The two versions of the model yield a similar prediction: if the government is in debt, it is optimal to issue debt in both short and long maturity. In fact b_t^S and b_t^N tend to have the same sign. Forcing the government to hold its obligations to maturity yields the prediction that debt is not entirely long term, as was the case under the buyback model.

[Figures 8 and 9 About Here]

The results in section 6.2 provide intuition for this model behavior: issuing long bonds is nice because it provides fiscal insurance, but short bonds are also nice because they avoid lumpy cash payments and, therefore, lumpy taxes. The optimal policy balances the two objectives by issuing (or purchasing) both types of bonds at the same time.

6.3.1 Model vs. Data Moments

In Table 3 we evaluate the performance of the no buyback model. Not surprisingly the moments shown under the 'Lending' model fit the average, standard deviation and serial correlation of the share very badly, the model has too much volatility of the share of short debt. This mismatch of the US data occurs because bonds switch sign often in the simulations as does the total market value of debt.³⁵ However the correlation of short and long debt is very high. This shows that b_t^S and b_t^N now move together as we had anticipated.

When the 'No Lending' constraint is introduced ($\underline{M}_S = \underline{M}_N = 0$) optimal policy is much closer to the empirical observations. We see in Table 3 that the model now delivers a substantial serial correlation (0.88 vs. 0.91 in the data) and a moderate volatility (0.085 vs. 0.06 in the data) in the share of short bonds. The correlation of short and long debt is very high. Therefore, assuming no

³³Strictly speaking the cycle in taxes would be less marked for shocks which are mean reverting (and in the case $\rho_g = 0.95$ which is our baseline), since in the intervening periods there would be a decrease of taxes, but an element of a reverberating shock would occur also in this case. The cycle would not occur in the case of permanent shocks, but at the same time it is meaningless to talk about debt management even from the point of view of the complete market-fiscal insurance analysis. No matter the structure of bond markets in this case, the tax rate must increase forever and fully finance the entire the rise in government spending if the shock is permanent.

³⁴More precisely: if \underline{M}_S would be sufficiently negative one could implement the complete market solution by saving in short bonds in the intervening periods and selling these short bonds every N periods to pay for the interest of the long bond. In the stochastic general model, however, these strategies are unfeasible and they can be eliminated by very loose negative \underline{M}_S .

³⁵As was the case with the buyback model, we find that under no buyback the planner accumulates precautionary savings in the long run. However, now the savings are not only held in short maturity, but in both types of bonds.

buyback enables us to match the data moments reasonably well. For the first time in this paper and, indeed in the literature, a model of optimal debt management matches the basic Facts 1-4, as well as Facts 5-7.

The fit is not perfect. The no buyback model overpredicts the average of the share relative to the US data (0.54 in the model vs. 0.36 in the data). We are not too concerned by the fact that we do not match this model implication, many issues are left out of our simple setup, we cannot fully capture the entire range of risks that governments face and which may induce debt management authorities to issue more long maturity debt. Furthermore, the average share is not very precisely estimated and it is highly model dependent, suggesting that a range or reasonable and minor modifications would bring this statistic in line with the data.^{36, 37}

Note that the positive correlation of short and long debt (Fact 4) is not driven by the no-lending constraint, this shows that the reason for matching this fact is the introduction of no buyback. Given this positive correlation it is not surprising that the share of one year debt is always positive in the simulations, therefore it matches Fact 2 much better than the model of section 5.1.

7 Extensions

7.1 Coupons

We have maintained the assumption that government bonds pay no coupons. Although this is the standard assumption in the literature most governments bonds do pay coupons. Coupons are not a relevant issue under complete markets but they certainly matter under incomplete markets. In particular, since coupons can be considered a form of debt at a shorter maturity they could matter for the prediction that short debt should be issued. We now assume that long term bonds pay out coupons every period. However, we continue to assume that the principal is redeemed at maturity and therefore we remain in a world with no buyback. We show that the analysis in Section 6 applies to this case.

When the government issues a bond it commits to the amount and the timing of the coupon payments. The size of the coupon may vary across issuance dates t . We denote by κ_t the coupon payment promised by the long bond issued at t . The bond then pays a constant amount κ_t from periods $t + 1$ to period $t + N$ and it also pays the principal (normalized to unity) in period $t + N$.

It is easy to show that the competitive equilibrium price of this bond is

$$q_t^N = \kappa_t \sum_{j=1}^N \beta^j E_t \left(\frac{u_{c,t+j}}{u_{c,t}} \right) + \beta^N E_t \left(\frac{u_{c,t+N}}{u_{c,t}} \right)$$

i.e. the price is now the sum of prices of zero coupon bonds of maturity $j < N$ ($p_t^j = \beta^j E_t(\frac{u_{c,t+j}}{u_{c,t}})$)

³⁶Within our model it is possible to bring down the average share by increasing the maturity of the long bond. Since the share is defined as all outstanding debt which is of one year maturity (including long bonds issued in the past and which have one year to mature), higher N gives a smaller fraction of long debt issued in the past which is nearing maturity. Through increasing maturity to $N = 18$ we produce an average share of 0.46 .

³⁷The literature has identified various reasons why long bonds may be preferred. For example, Greenwood et al. (2010) add further shocks to the economy including a shock to the discount factor, which would give rise to variability in interest rates and debt refinancing risks. In this case the government would like to ward off these risks through issuing more long maturity debt. The clientele narrative in Guibaud et al (2013) may also be used to increase the share of long bonds if a 'long-horizon clientele' is added to the model.

weighted by the coupon payments promised by the bond. Moreover, in the steady state (with constant coupons and consumption) the bond price becomes equal to $\kappa \sum_{j=1}^N \beta^j + \beta^N$.

According to the CRSP data, bonds issued by the US government trade close to par when they are issued, namely $q_t^N \approx 1$. To choose coupons that are consistent with this observation we set $\kappa_t = \kappa = \frac{1-\beta}{\beta}$ for all t , where β is the utility discount factor. It turns out that in this case bonds trade close to par, our simulations yield q_t^N close to 1 in all periods.

7.1.1 The Ramsey Program under Non-Zero Coupon Bonds

We now find the optimal policy assuming that long bonds pay a yearly coupon κ_N . Debt limits are:

$$b_t^N \in \left[\frac{\underline{M}_N}{\sum_{j=1}^N \beta^j + \kappa_N \sum_{j=1}^N \sum_{i=1}^j \beta^i}, \frac{\overline{M}_N}{\sum_{j=1}^N \beta^j + \kappa_N \sum_{j=1}^N \sum_{k=1}^j \beta^k} \right] \equiv [\widetilde{M}_N, \overline{\widetilde{M}}_N]$$

and $[\widetilde{M}_1, \overline{\widetilde{M}}_1] \equiv [\frac{M_1}{\beta}, \frac{\overline{M}_1}{\beta}]$ for short bonds.³⁸ The planning problem and the FOC with respect to consumption are given in the online appendix.

Off corners the first order conditions for b_t^1 and b_t^N are:

$$(30) \quad \lambda_t E_t(u_{c,t+1}) = E_t(\lambda_{t+1} u_{c,t+1})$$

$$(31) \quad \lambda_t E_t(\kappa_N \sum_{j=1}^N \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t(\kappa_N \sum_{j=1}^N \beta^j u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N}).$$

7.1.2 Repayment Profiles and Optimal Portfolios under Non-Zero Coupon Debt

The previous section illustrated that when the government cannot buy back its debt before maturity, long term zero coupon bonds feature a lumpy redemption profile. This feature, in the context of the Ramsey problem was shown to be linked to the fact that the dynamics of λ_t were characterized by an N period cycle.

Equation (31) is the analogue of equation (25) (when the debt constraints are loose) for the no-buyback model with positive coupons. It reveals that the multiplier follows a complicated pattern which equates it with the sum of all expected future terms $u_{c,t+j} \lambda_{t+j}$ for $j = 1, 2, \dots, N$ weighted by the payments that the bond promises.

[Figure 10 About Here]

To illustrate the effects of a fiscal shock, we study simulations of this model with the same calibration as before. In Figure 10 we show the response of the tax schedule to the spending shock. The main implication is that under coupons the optimal tax schedule features a similar pattern as under no buyback (previous section) characterized by an important N cycle component.³⁹ We conclude that under positive coupons the main features of optimal fiscal policy under no buyback persist, the interest rate payments are still concentrated every N periods.

³⁸In the US coupon payments are usually six monthly. However, since our model's horizon is one year we model one year debt as zero coupon.

³⁹The difference from Figure 7 is that we now assume that shocks are persistent as in our baseline calibration. We obtain a similar profile when we set $\kappa_N = 0$.

In Figure 11 we show a sample from the optimal portfolio model under ‘No Lending’ and with coupons. As is evident from the figure the basic implication is that the government uses both short term and long term debt. The resulting debt sequence is strikingly similar to that of the zero coupon model. Moreover, the last row of Table 3 shows that the key moments in the non zero coupon model are nearly identical to those of the benchmark no buy back model, featuring only a small improvement in terms of the volatility and the serial correlation of the share.⁴⁰

[Figure 11 About Here]

7.2 Buybacks/Callable Bonds

We now illustrate that our assumption of no buyback may be further relaxed to let the government purchase long term debt prior to maturity. In particular, we show that if the long term bond of maturity N is removed from the market $m < N$ years after its issuance, the optimal portfolio will still feature a large share of short maturity (one year) debt.

As was discussed in Section 2 a substantial fraction of long term debt in the US in the 50s and the 60s was held in callable bonds (details are section A.2 of the Appendix). The government very frequently exercised the option: in roughly 75% of all cases the Treasury removed from the market callable bonds at the first possible call date. Recall, also, that the first call date is quite close to maturity. For 10 year notes the first possible call date is between 2 and 5 years before the bond matures, for 25 year bonds the first callable date is 5 years before maturity, for 30 year bonds it is between 5 to 10 years. (see Table 2).

Rather than setting up the planner’s program giving the explicit option to buyback, we assume that a bond issued in t is completely bought back at $t + m$, since the data tells us that this is a reasonable approximation to the realized debt management strategy. We set $N = 10$ as previously, and also $m = 5$. Notice that given the actual callable dates this chooses m to be the earliest possible call date observed, thus making these long bonds as close as possible to buyback.

Off corners the first order condition with respect to b_t^N (see our derivations in the online Appendix) is given by :

$$E_t(u_{c,t+N}\lambda_{t+m} - u_{c,t+N}\lambda_t) = 0$$

and therefore as in the case of the no buyback model of Section 6 if we ignore short bonds the multiplier λ_t has an m -year (rather than an N year) cycle.

Figure 12 presents our simulations of the ‘No Lending’ model. The model behaves in a similar way as in the previous cases.

[Figure 12 About Here]

We conclude it is not necessary to impose that the government purchases long term debt at maturity for our results to go through. No buyback maybe more loosely interpreted as keeping debt in the market for sufficiently many periods after its issuance.

⁴⁰When we assume ‘Lending’ our results are similar to the ones obtained in section 6. We omit them for sake of brevity.

8 Summary of the Empirical Results

The object of the paper is to explore to what extent a standard Ramsey policy model can match basic observations on debt management. We claim that there is a huge gain from introducing no buyback, and that 'No Lending' also helps. Not surprisingly we do not match closely all the moments, for example the sample mean of the share of short bonds $S_{1,t}$, see Table 3.

Looking only at Table 3 may leave the reader undecided about goodness of fit, after all no model matches the sample moments perfectly. We now provide some discipline as to whether a moment of a model is "sufficiently close" to the sample moment. This will also tell us where there is room for improvement. We compute t-statistics for the null hypothesis that the population statistics reported in Table 3 are equal in the data and in the models. We do not estimate parameters so as to improve the fit, we keep the parameters from Marcet and Scott (2009) who did not consider debt management.

To make t-statistics comparable across models and for robustness, we estimate the standard deviation of each statistic considered in Table 3 from the data.⁴¹ The results are shown in the first row of Table 4. The t-statistics are found dividing the difference between each statistic in the data and in each model by the standard deviation of each sample statistic. The results are shown in rows 3 to 7 of the Table. The t-statistics have a standard normal distribution asymptotically.

As can be seen the model with no buyback, no lending and coupons matches all moments considered except the mean of $S_{1,t}$. In particular, note how all the t-statistics in the last row of the same Table (except for the mean) are within the 95% confidence interval.

We find this surprising. First, because the models under buyback (closest to the available Ramsey debt management literature) have huge t-statistics, even the buyback model with lending does very badly. Second, because we did not set out to match empirically all these moments, were only looking for qualitative results using standard parameter values and standard Ramsey models.

The only statistic we do not match is the mean of $S_{1,t}$. We do not see this as a crucial failure at this point. The previous literature using Ramsey models of debt management with buyback found that short government bonds issuance was negative or equal to zero. Even our incomplete market model under buyback and no lending (section 5.2) shows a very low average share, and $S_{1,t} = 0$ in many periods, while this never happens in the data. We conclude that no buyback matches qualitatively the observed positive issuance of short bonds although it does not match it quantitatively.

Many reasonable modifications of our model should help in lowering average $S_{1,t}$ and make it closer to the data. One could introduce a role for long bonds that is not captured in the current paper. Some alternatives are: introducing a bond market clientele, or rollover risk, or the role of long debt in tying the hands of rival parties in future governments, etc. We do not pursue these in the current paper.

⁴¹To estimate the standard deviation of sample averages we use the fact that, for a mean zero stationary and ergodic process x_t we have $\frac{\sum^T x_t}{\sqrt{T}} \rightarrow N(0, S_w)$ in distribution. We estimate $S_w = \sum_{j=-\infty, \infty} E(x_t x_{t-j})$ using the Newey-West statistic. The asymptotic standard deviation of functions of moments (such as correlations) are found with the delta method.

9 Conclusion

The literature on debt management has difficulties in explaining basic features of observed government bond portfolios. It has often produced recommendations that were extremely different from observed debt management in most countries.

In this paper we have sought to throw light on why governments issue short term debt as well as long term debt in substantial amounts, a combination that the canonical Ramsey model of optimal fiscal policy finds challenging to explain. Furthermore we explain why the level of short debt correlates positively with total debt, the volatility of short debt. We do so by introducing a feature of actual debt, namely, that long debt is held until maturity, or recalled shortly before maturity. When governments have to hold debt to maturity the timing of cash flows matters, long bonds concentrate cash payments on a few periods, giving governments a desire to use short term debt so as to spread interest payments and debt repayment more evenly over time. Short term bonds provide cash flow benefits whilst long bonds provide better fiscal insurance, as has been previously identified in the literature. Therefore, optimal debt management implies issuance at many maturities and all these should move together with total debt.

We show how this result is immune to the introduction of callable bonds and also helps explain why governments like to issue coupon paying bonds.

Making the assumption that bonds are held to maturity substantially complicates the state space of our model and so requires two significant algorithm features (Forwarded- and Condensed-PEA) that may be useful in a variety of dynamic models.

We have focused in this paper on showing the importance, both empirically and theoretically, of the fact that governments tend not to buy back debt until its maturity. We have shown how under incomplete markets making this assumption matters as it affects time timing of cash flows for the government and has a substantive effect on debt management. The result is that the predictions of the canonical Ramsey model move dramatically closer to those we observe in the data and help to explain a variety of real world features of debt management. However much work remains to provide a full understanding of debt management not least of all endogenising the no buyback feature. Our approach also needs to be extended to include refinancing risks as well as preferred habitat, clientele models.

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A Data Appendix

A.1 Database and Construction of Figures

Data on US Treasuries was obtained from the CRSP US Treasury Database and is comprised of all types of marketable treasury securities (bills, notes, bonds and inflation protected securities (TIPS)). Observations are available at a monthly frequency and were taken from the period 1955-2011. Date variables of particular interest for this study included the quote date, the date of the first coupon and the maturity date. Amounts outstanding of the bonds were usually available although missing for certain observations. Gaps in amounts outstanding were filled with preceding observations when available for the same bond or future observations if no preceding data points exist. Coupons are typically paid every six months from the date the bond is issued and until maturity.

To construct the share of short term debt (in Figure 1) we stripped the coupons. The strips, were given distinct maturity dates, face values and market values. For a ten year bond paying coupons every six months the first coupon is counted as six month maturity debt (at the issuance date), the second coupon as one year debt and so on.⁴² Market values for the strips were imputed using the yield-to-maturity data (available from the database as a one-day rate) for the principal bond from which they were separated. The parent bond's price outstanding was adjusted by subtracting the present value of its coupons.

The share of short term government debt which is plotted in Figure 1 is comparable to Greenwood et al. (2010, Figure 3). The difference between our approach in constructing the sample and theirs, is that we include inflation protected debt in our calculation. TIPS typically represent long maturity claims (typically five, ten or thirty year debt) and therefore, in our case, contribute towards reducing the average value of the share of short term debt. This is so in the post 1997 part of our sample when the first inflation protected security in the US government bond market was issued (before then there was no indexed government debt).

Finally, to construct the time series of the issuances (Figure 2) we adopt the following approach. Because our definition of short term debt in the data includes maturities which are less than one year (one month to six months in the CRSP data) we define the total issuance in short maturity debt as : $I_{S,t} = \frac{I_{1m,t}}{12} + \frac{I_{3m,t}}{4} + \frac{I_{6m,t}}{2} + \frac{I_{12m,t}}{1}$ where $I_{xm,t}$ denote claims of maturity x months in the data⁴³ and the quantities $I_{xm,t}$ are sums of issuances over year t . Note that differently from the case of the stocks displayed in Figure 1 the issuances share is formed by a flow variable in the numerator and a stock variable in the denominator. This approach enables us therefore to not bias the share upwards by counting very short term issuances (monthly debt) many times over the year.

A.1.1 Bond Prices for Non-Zero Coupon Bonds

As discussed in the text bond prices of non zero coupon bonds typically float near par value (i.e. at a market price equal to the principal amount) but rarely trade exactly at par. In Figure 13 we

⁴²Our approach to strip the coupons and assign an appropriate maturity to each strip is compatible with the notion that the benefits from fiscal insurance are proportional to the amount of long term government debt outstanding. If a long term bond is issued paying coupons every six months then this bond should provide less insulation to the governments budget than a zero coupon long term bond. When we construct the model counterparts for the series plotted in Figure 1 we follow essentially the approach.

⁴³This definition includes the first coupons of long term notes and bonds.

provide evidence supporting this claim through plotting the percentage deviation of the price from par for all bonds which pay coupons in the CRSP dataset. The data concerns issuances in the post 1955 period and therefore omits long term debt which has been issued during WWII or shortly after, and during the Korean war. As the figure shows for the vast majority of long term debt issued, bond prices at the issuance date lie within the interval $[-5\%, +1\%]$ of par value (roughly 40 percent trade exactly at par value). This is essentially the claim we made in section 7.1.

[Figure 13 About Here]

A.2 Callable Bonds in the US

In Figure 14 we illustrate the percentage of callable securities over the total value of long term debt outstanding in the US. Callable debt is basically long maturity government debt which embeds an option to redeem the principal (at par) at specific dates prior to maturity. These call dates are essentially the coupon dates of the bond but they are active past a certain period from the bonds issuance. For example a ten year bond issued with a call option gives the right to the treasury to redeem the amount of debt outstanding at any coupon date 5 years before the bond matures. Therefore, this bond can be withdrawn from the market at the earliest five years after its issuance. Call options are obviously embedded in longer term debt (for example 30 year bonds) and in this case the first call date may be earlier than 5 years prior to maturity (for instance 10 or 15 years).⁴⁴

[Figures 14 and 15 About Here]

As Figure 14 shows a substantial fraction of long term government debt is callable. In the 50s up to half of overall long term government debt outstanding embeds a call option with the fraction falling over the sample period. Since 1985 most of the issues of long term debt have been non-callable (though it is always possible to add a call feature via derivatives created by non-government issuers), explaining the small fraction in the latest part of the sample.

A.2.1 Call Windows For Callable Bonds in the US

In Table 5 we summarize the data on callable debt along the following dimension: with each row we represent the maturity of the bond and with each column the call window, the difference between the first possible call date and the maturity. For example, the first row which refers to 5 year maturity debt at the date of the issuance, shows that all of the three 5 year bonds we encounter in the data have a call window of two years. These bonds may be withdrawn from the market two years prior to maturity, or, equivalently, three years after the issuance date. Analogously, the last row of the table shows that most of the debt issued in thirty year maturity has a window of 5 years, meaning that call options are active 25 years after the issuance date. Overall, the evidence in the table suggests that for debt of 25 to 30 year maturity, call windows may be 5, 10 or even 15 years. For the case of shorter maturity debt the bonds are typically redeemable at 2 or three years before the maturity date.

⁴⁴The difference between the market prices of a callable treasury and a non-callable equivalent reflects the call option value. Thus a callable bond should, in theory, never sell at a higher price than a non-callable equivalent. In practice, however, it is not uncommon to observe callable securities being overpriced, with the implied option values being negative (see Longstaff (1992)).

A.2.2 Are Call Options Typically Exercised?

We investigate whether call options on long term debt are typically exercised in the historical observations and provide information concerning the redemption profile (timing) for these bonds.

Table 2 lists the issuance dates of callable debt in the US, the total amounts issued in such securities (1st column) and the fraction of the amount outstanding at every issuance which has been withdrawn from the market (redeemed prior to the maturity date). Note that aside from a few cases in the late 50s and the early 60s it is typical for all callable debt to be bought back, i.e. for the government to exercise the option to redeem it at one of the call dates before the bond matures.

As discussed previously, given the structure of the call option the Treasury has the possibility of redeeming callable debt on every coupon dates (i.e. every 6 months) after the first predetermined call date. For example bonds which have a window of 2 years (see Table 5) give the right to the issuer to buyback the debt outstanding 2 years before maturity, then at 1.5 years then at 1 year and so on. In order to illustrate how US debt management has historically exercised the option to buyback long term government debt which is callable we show in Figure 15 the distribution of the timing of redemptions given the call window. The top left panel shows the fraction of callable debt with a two year window which is redeemed on the first call date (-2 years), between 1 and 2 years (-1 years) and in less than one year from maturity (0 years). The top right shows the same for the case of bonds with a three year window, the bottom left for four year window and the bottom right for five year.

As the figure shows, the fractions of callable debt by different call windows which have been bought at the first possible call date is substantial. For example, the top left panel of the Figure illustrates that roughly 75% government debt with a call window of two years is bought at the first call date. Similar fractions of debt outstanding have been bought in the case of 3, 4 and 5 year window bonds. For the remaining 20-30% of callable bonds, buybacks occur either 1 year before maturity or close to it (the value zero summarizes redemptions which happen even 6 months prior to maturity).

We conclude that callable bonds have indeed been a substantial fraction of long term debt in the US at least in the early periods of our sample. They typically are issued with a call option which is active several years after the issuance date and they are redeemed at the first possible call date. In this paper we show that in theory the no buyback assumption is a good approximation for this type of debt provided that the first call date is at least two years after the issuance of long term debt (as is the case in the empirical observations).

A.3 Redemption Profiles for Non Callable Long Maturity Debt

As we explained in Section 2 non callable long term debt (which forms a substantial part of the governments portfolio) is typically held until maturity. The evidence is discussed extensively in Marchesi (2006) and Bloomstein et al (2012) for the OECD countries. According to their analysis buybacks of long term debt occur but occur typically very close to maturity (less than one year before the long term bond matures) and are aimed at smoothing redemption profiles.

We wish to provide here additional evidence to support this finding, focusing on the US debt management and extending the analysis of Bloomstein et al (2012) to document no buyback over

lengthier sample (history) of observations. With the CRSP data we study all non callable long term bonds issued since the 1920s and in Figure 3 and Table 1 we show the redemption profiles by quarters. According to the results in Table 1 roughly 93.6% of all long maturity government debt was redeemed at maturity (0 quarters) and another 2% is bought back at some point up to one year before maturity. Figure 3 summarizes these profiles in a histogram.

Note that in our sample we have included observations since the 1920s to account for two major buyback episodes covered by the CRSP database. The first is the buyback during the 1920s and the second is the buyback of long term government debt in the early 2000s. Our reading of these episodes is the one from Garbade and Rutherford (2007) and (for the buyback in 2000-2001) from Greenwood and Vayanos (2010). It is well understood that on both occasions the Treasury, due to large and persistent budget surpluses in the 20s and the late 90s, bought back long term government debt to avoid running down issues of short maturity notes.⁴⁵ However, as Garbade and Rutherford (2007) explain buybacks is not a tactical feature of debt management. This conclusion is in line with the results in Table 1 which provides strong support for no buyback in the case of non-callable long term debt.

[Table 5 About Here]

⁴⁵Garbade and Rutherford (2007) document that this is precisely what happened in the 90s as a result of the Treasury's standard practice to not buyback long term bonds before maturity. The persistent surpluses convinced the authorities to consider buybacks of long bonds. The buyback defined as a consensual transaction between the Treasury and the private sector, involved purchases of the following securities: 1. callable bonds maturing between February 2010 and November 2014 (this consists of very long term debt issued in the 80s), 2. non-callable bonds maturing between February 2015 and 2019, 3. non-callable bonds maturing between 2019 and 2022 or 2023, and 4. non-callable bond maturing between 2022 or 2023 and November 2027. As Garbade and Rutherford (2007) document between 2000 and 2001 roughly 14% of long term government debt (in market value) was cancelled as the result of the buybacks operations.

Remaining Term (in quarters)	Normalized Count
0	93.63%
1	1.06%
2-4	1.00%
5-9	0.83%
10-14	0.75%
15-29	1.58%
30-45	0.37%
>45	0.78%

Notes: The table provides information on the redemption profiles of non callable bonds. The data are all non callable debt issued by the US Treasury since the 1920s. Remaining term counts the number of quarters remaining until maturity when debt is bought back. When 0 this signifies that debt is bought at maturity. The data are extracted from the CRSP.

Table 1: **Remaining Term at Time of Buyback**

Year	Amount Issued (in millions)	Share Called
1931	755	100%
1934	491	100%
1935	2611	100%
1936	5616	100%
1938	3588	100%
1939	1689	100%
1940	1404	100%
1941	9326	55.35%
1942	14061	38.06%
1943	16763	29.46%
1944	26986	14.16%
1945	28172	11.71%
1952	921	100%
1953	1606	0%
1960	470	0%
1962	365	0%
1963	550	54.55%
1973	1618	100%
1974	587	100%
1975	3616	100%
1976	1574	100%
1977	2638	100%
1978	4516	100%
1979	4523	100%
1980	7794	100%
1981	4626	100%
1982	3163	100%
1983	4921	100%
1984	16142	100%

Notes: The table lists (by year of issuance) the total amounts of callable bonds which have been called prior to maturity. The data are extracted from the CRSP.

Table 2: **Share of Redeemed Callable Treasuries by year of issuance**

	\bar{S}_1	σ_{S_1}	$corr(S_{1,t}, S_{1,t-1})$	$corr(\tilde{b}_t^S, \tilde{b}_t^N)$
US DATA	36%	0.06	0.91	0.786
BuyBack				
'Lend.'	272%	76.9	0.22	-0.473
'No Lend.'	26%	0.17	0.79	0.117
No Buy back				
'Lend.'	16%	8.63	0.33	0.762
'No Lend.'	56%	0.085	0.88	0.906
'No Lend.+Coupons'	53%	0.073	0.92	0.915

Notes: S_1 denotes the share of total debt of maturity less than or equal to one year over the total (market value of debt). \bar{S}_1 represents the average share and σ_{S_1} denotes the standard deviation. The statistic $corr(\tilde{b}_t^S, \tilde{b}_t^N)$ is the correlation between the market value of short debt and the value of long debt. The exact definition varies depending on the model specification. For example under buyback it holds that $corr(\tilde{b}_t^S, \tilde{b}_t^N) \equiv corr(p_t^S b_t^S, p_t^N b_t^N)$. Under no buyback and no coupons $\tilde{b}_t^N \equiv \sum_{i=S+1}^N p_t^i b_{t-N+i}$. Therefore, when $S = 1$ the value of long debt outstanding is the value of all debt which in t is of maturity greater than one year and \tilde{b}_t^S is the market value of all outstanding debt less than one year maturity. The data counterpart is constructed applying this logic (see text).

Table 3: Moments: Data and Models

	\bar{S}_1	σ_{S_1}	$corr(S_{1,t}, S_{1,t-1})$	$corr(\tilde{b}_t^S, \tilde{b}_t^N)$
std. of sample moment	0.026	0.006	0.026	0.096
Buyback				
'Lend.'	90.47	12739	-26.80	-13.13
'No Lend.'	-3.63	17.81	-5.02	-6.98
No Buyback				
'Lend.'	-7.45	1421	-22.60	-6.98
'No Lend.'	7.85	3.72	-1.57	1.24
'No Lend.+Coupons'	6.70	1.73	-0.05	1.33

Notes: The Table presents t statistics testing the hypothesis that the data moments are equal to the model generated moments summarized in Table 3.

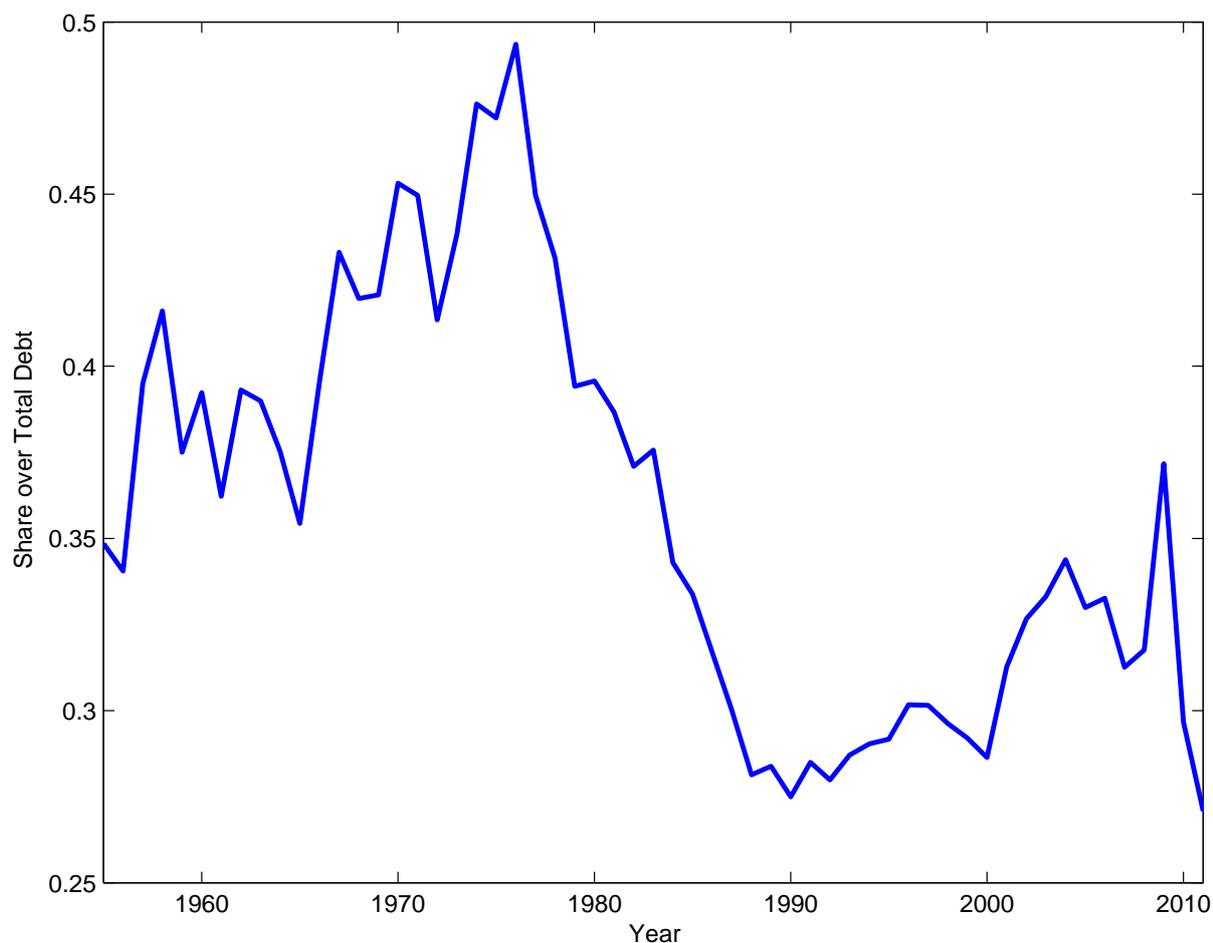
Table 4: t stats: Data and Model Moments

Bond Term (in years)	Call Window* (in years)					
	2	3	4	5	10	15
5	3	0	0	0	0	0
7	1	0	0	0	0	0
9	1	0	0	0	0	0
10	8	0	0	0	0	0
11	1	0	0	0	0	0
12	3	0	0	0	0	0
13	1	0	0	0	0	0
14	3	1	1	0	0	0
15	1	2	0	0	0	0
16	1	1	0	0	0	0
17	1	2	1	0	0	0
18	0	3	0	0	0	0
20	0	0	1	1	0	0
23	0	1	0	0	0	0
24	0	0	1	0	0	0
25	0	0	0	8	1	2
26	0	0	0	2	0	0
27	0	0	0	6	0	0
29	0	0	0	0	0	2
30	0	0	0	13	2	2

Notes: The table shows the call windows (maturity minus first possible call date) for callable bonds in the US. The data are extracted from the CRSP and refer to government debt issued since the 1940s.

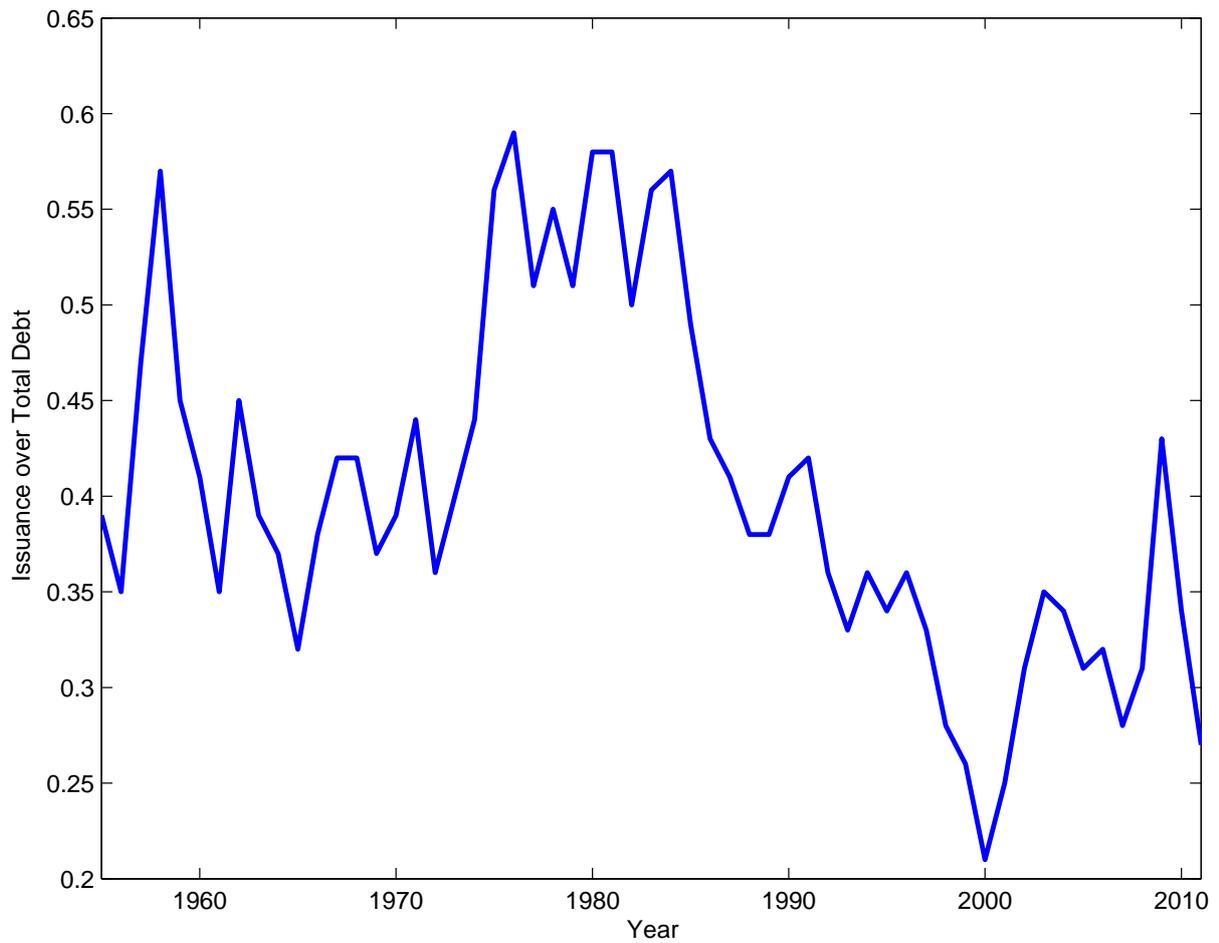
Table 5: **Bond Terms and Call Windows**

Figure 1: Share of Short Term Debt in the US



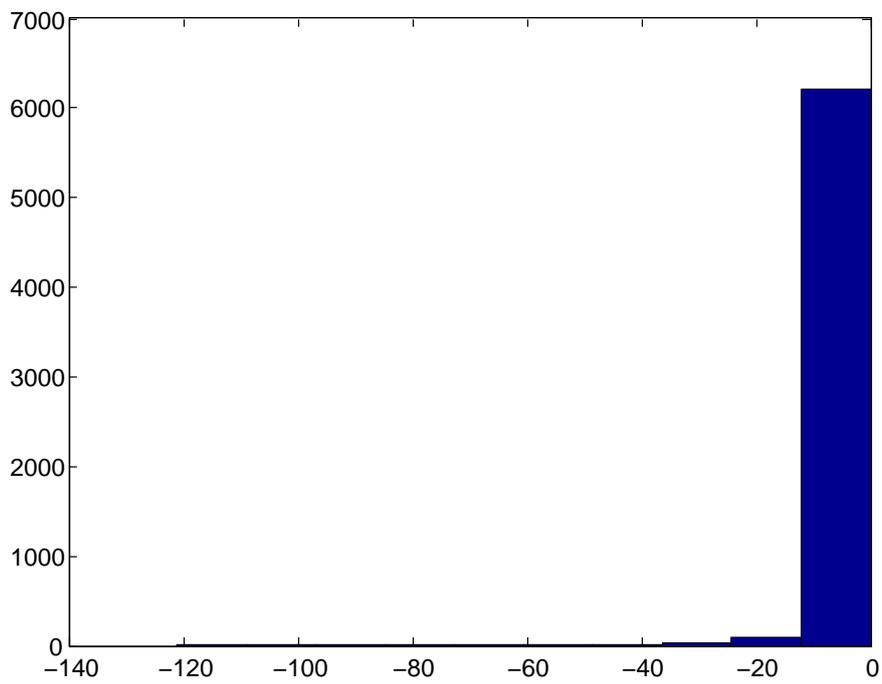
Notes: The Figure plots the share of short maturity government debt (less than or equal to one year) in the US over the period 1955-2011. The data are annual observations (time aggregated from monthly data extracted from the CRSP). Details on the data construction are contained in the Appendix.

Figure 2: **Total Issuance as a Fraction of the Market Value of Outstanding Debt**



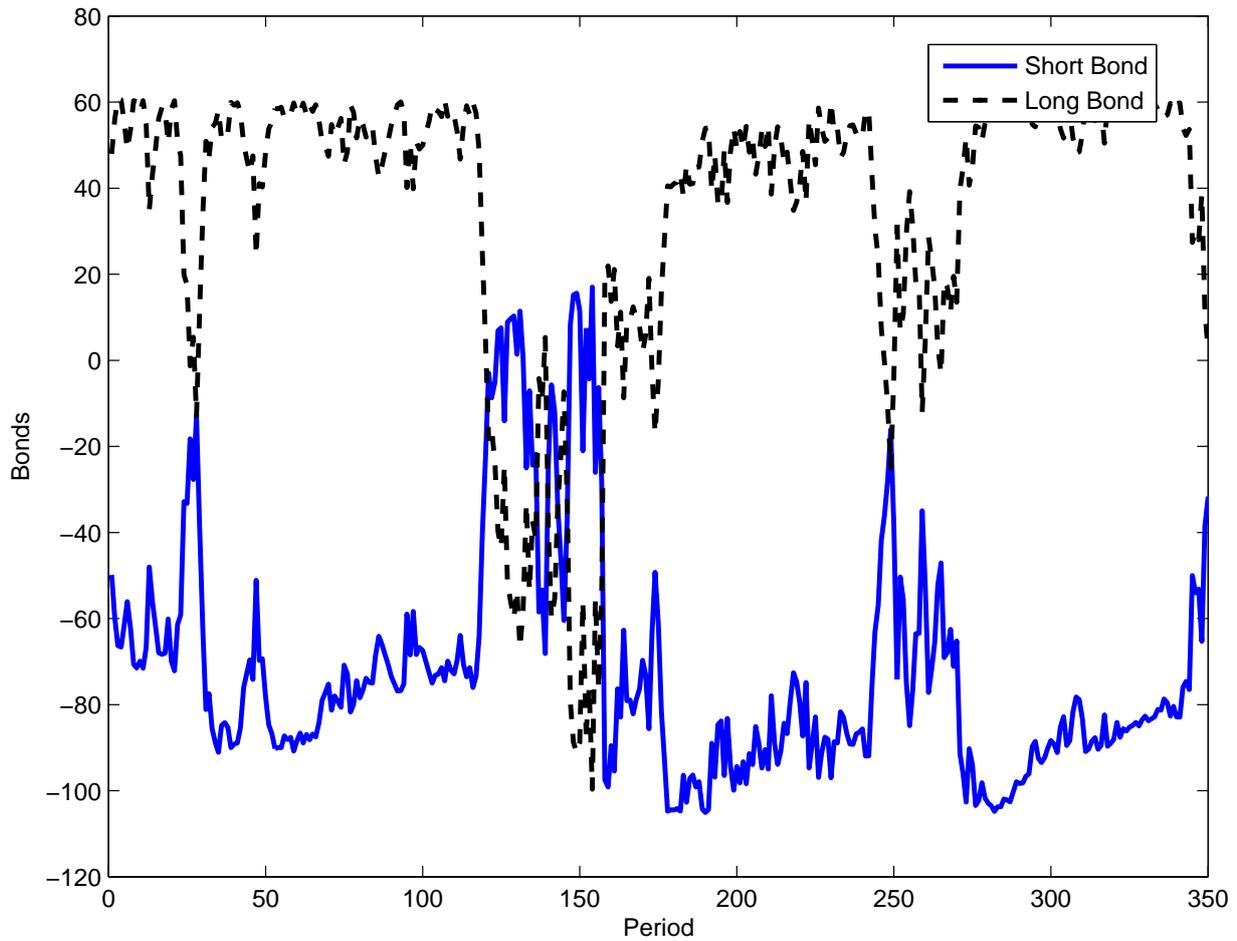
Notes: The Figure plots the issuance of new government debt by year in the United States as a fraction of the total market value of government debt outstanding. The data are from the CRSP and refer to the period 1955-2011.

Figure 3: **Redemption Profiles for Non-Callable Bonds**



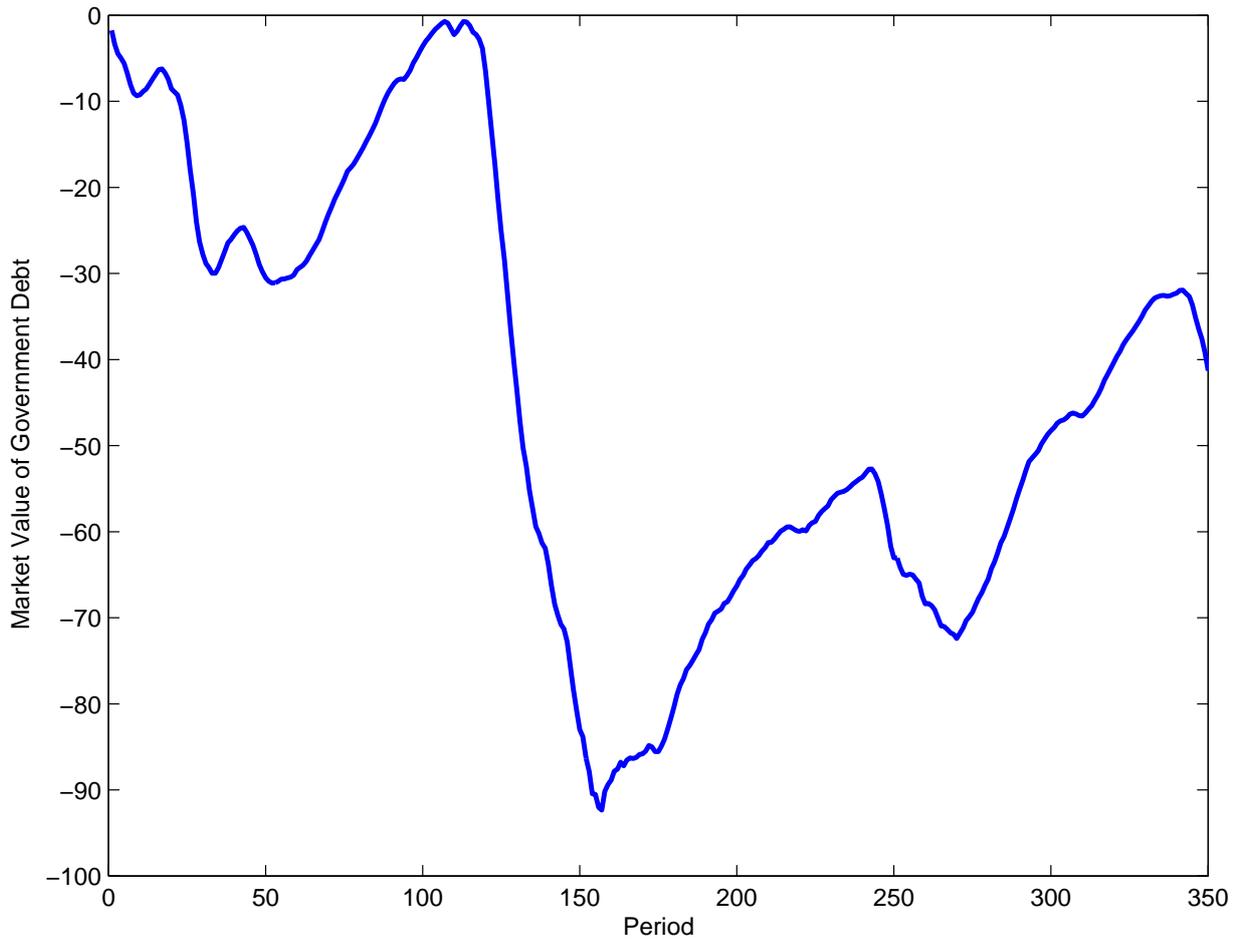
Notes: The plots shows the timing of buybacks for non-callable debt in the US. The data are extracted from the CRSP and corresponds to all issuances of long maturity government debt since the 1920s.

Figure 4: Optimal Portfolio under Buyback - Lending Model



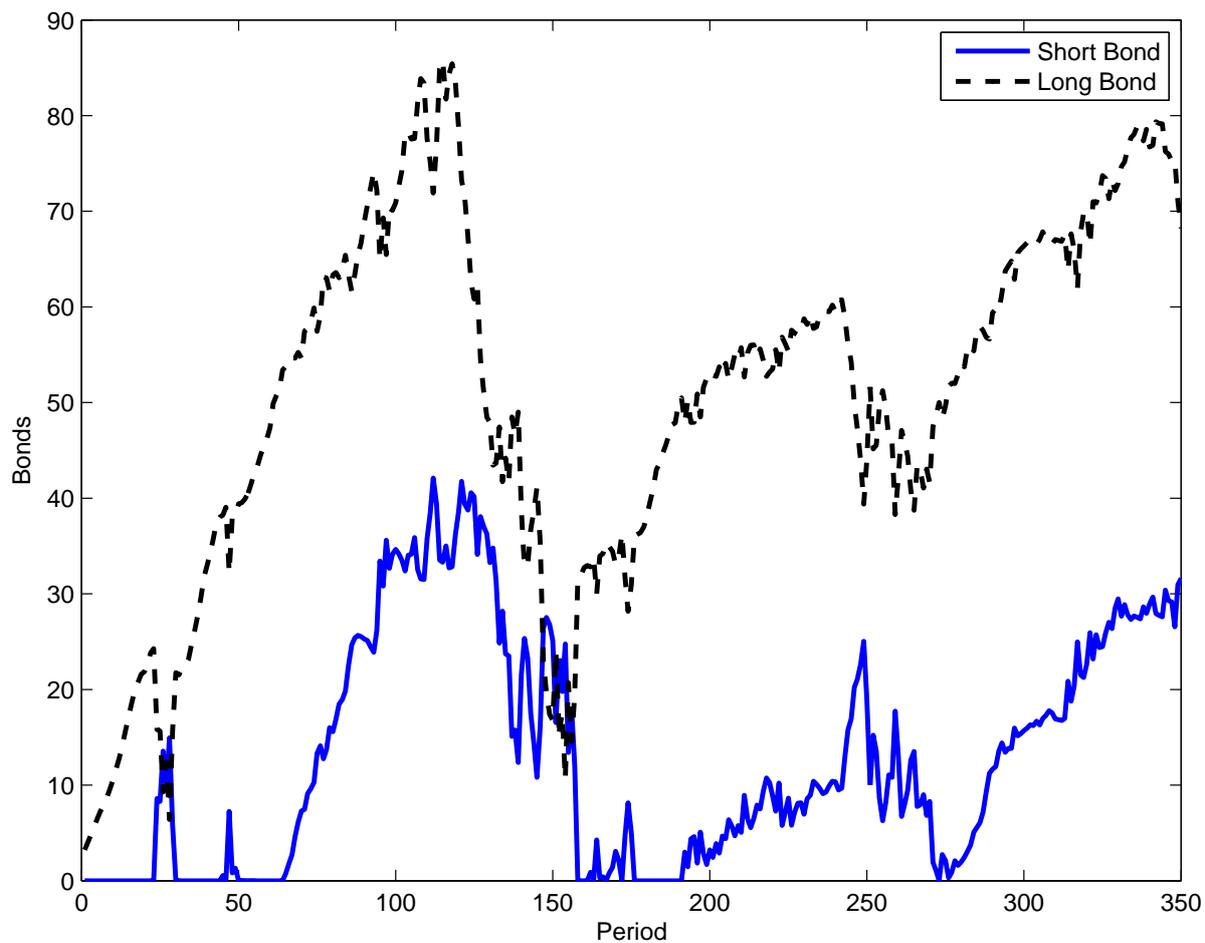
Notes: The Figure plots a sample path of the optimal portfolio under buyback. The bounds (upper and lower) correspond to 150% of (steady state) GDP. The solid line represents the short term bond. The dashed line the long maturity bond.

Figure 5: Market Value of Government Debt - 'Lending' Model



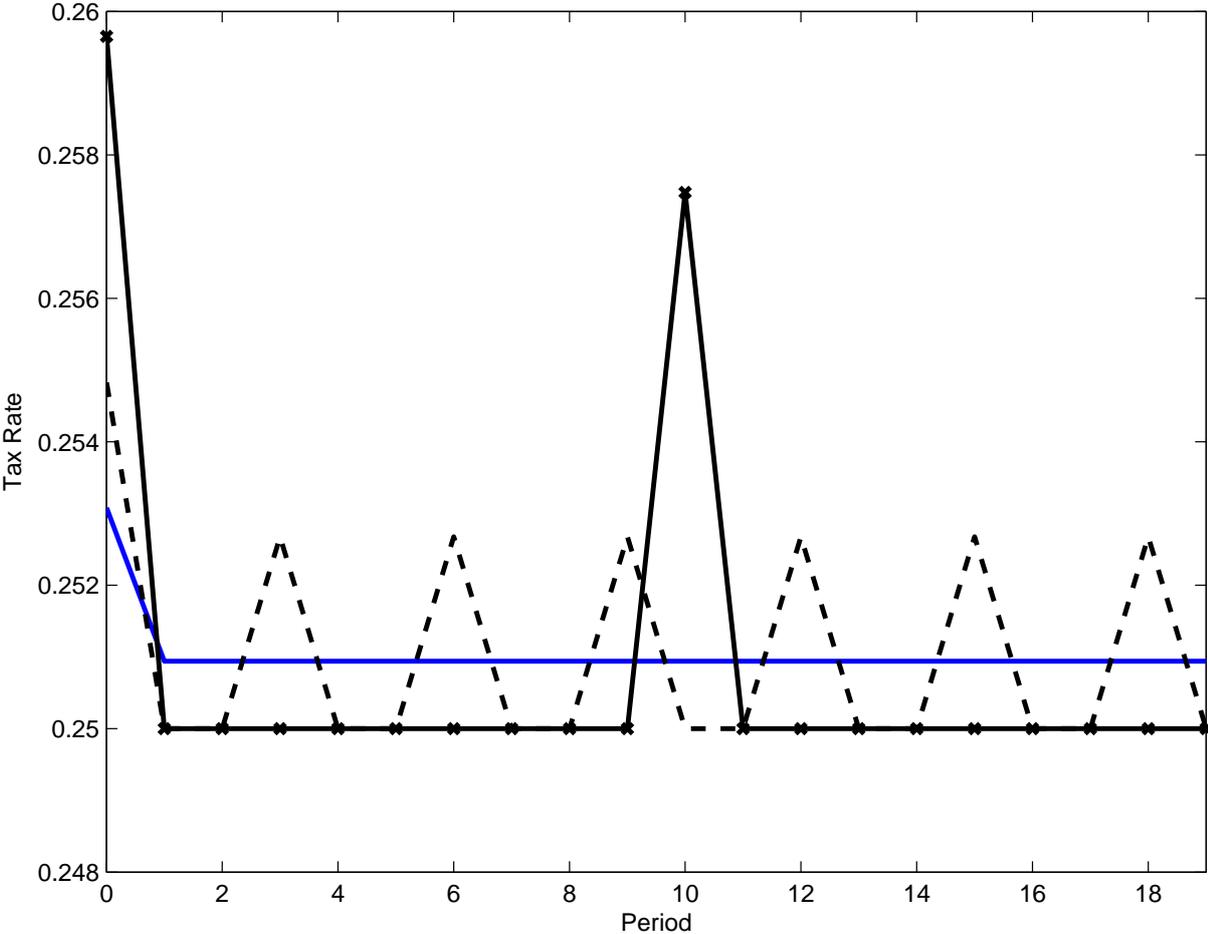
Notes: The Figure plots the market value of government debt under the buyback model. The portfolio (long and short bonds) which corresponds to the market value is shown in Figure 4.

Figure 6: Optimal Portfolio under Buyback - 'No Lending' Model



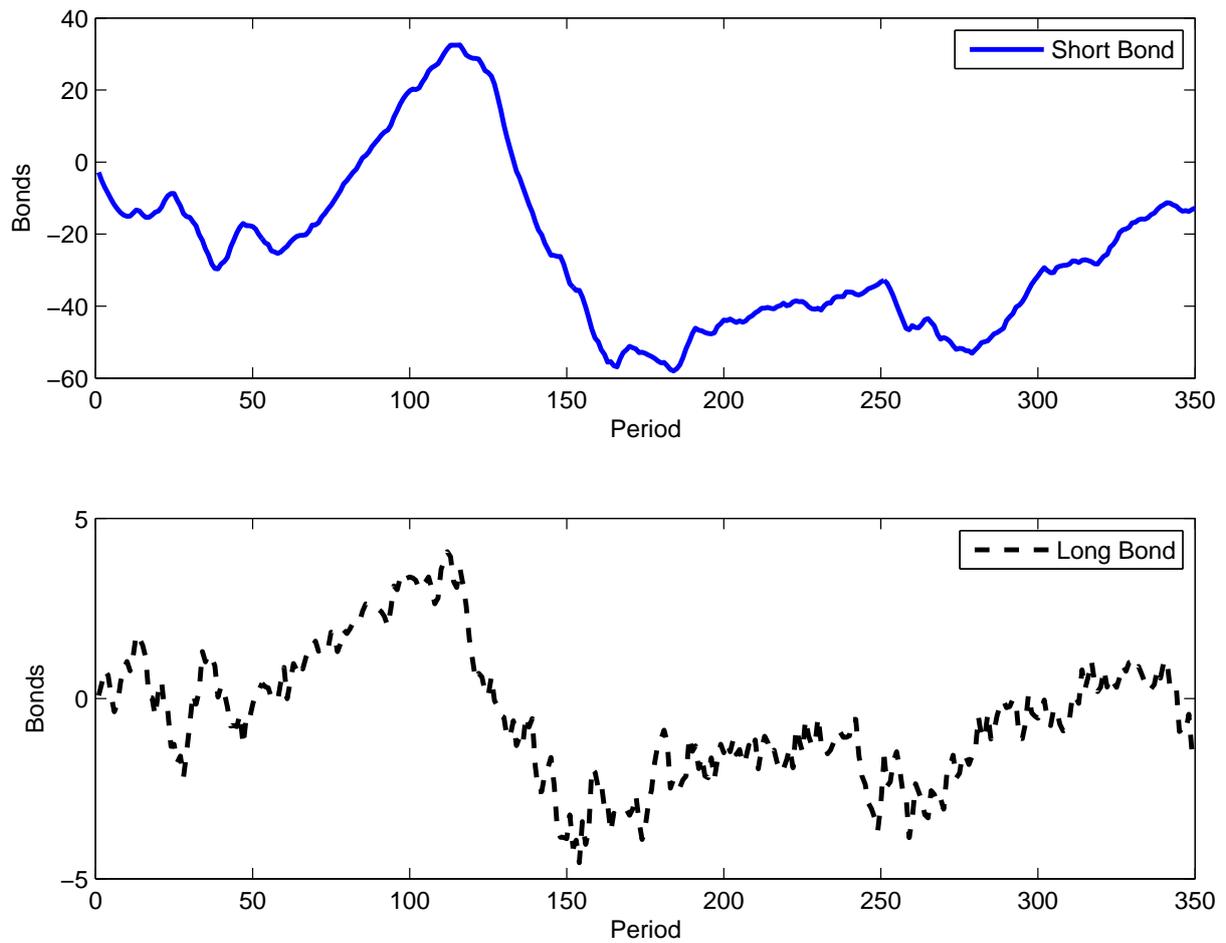
Notes: The Figure plots a sample path of the optimal portfolio under buyback and 'No Lending'. The upper bound is 150% of (steady state) GDP. The solid line represents the short term bond. The dashed line the long maturity bond.

Figure 7: Response of the Tax Schedule - No Buyback Model



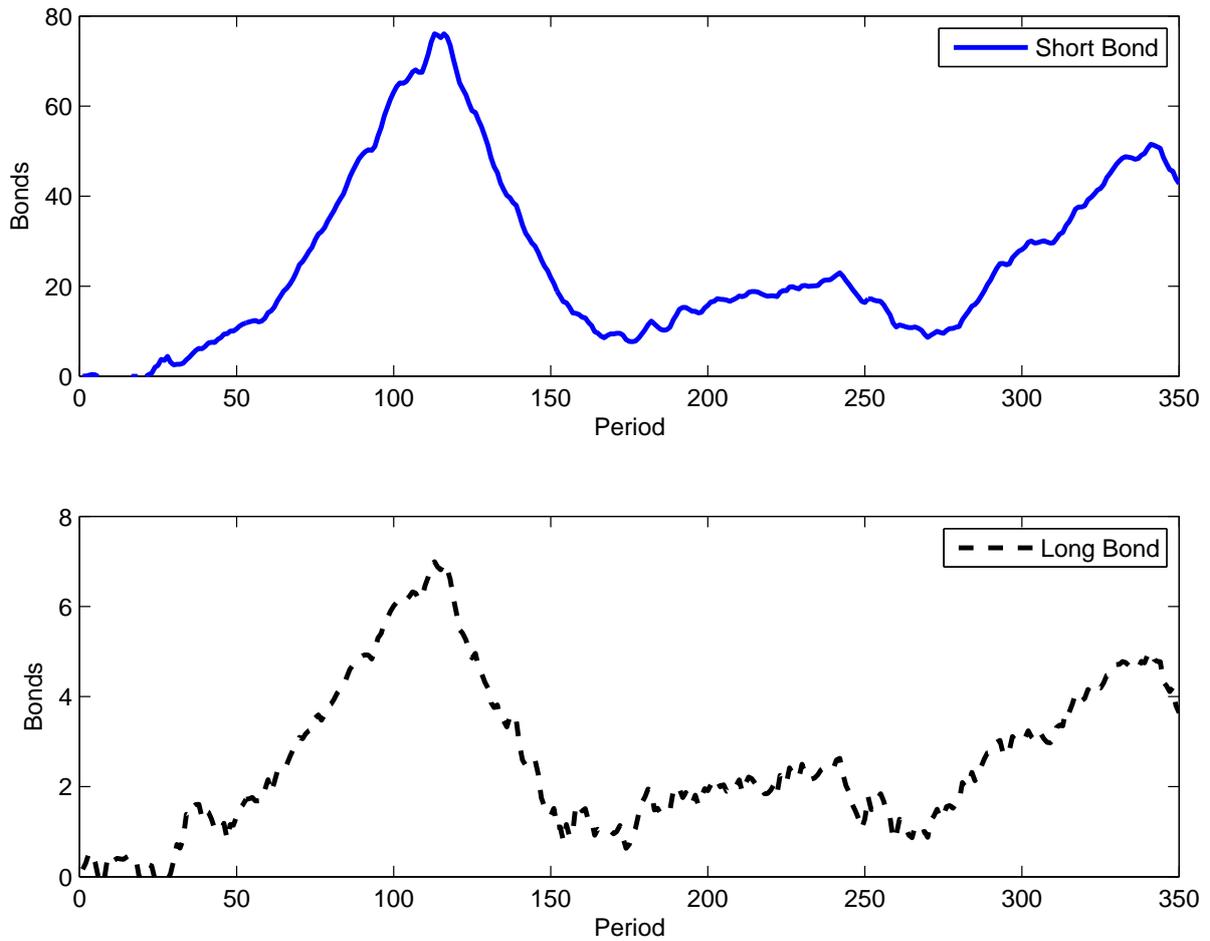
Notes: The Figure plots the tax rate in a single bond economy as in the example of section 6.2.2. The solid line is for a maturity of one year, the dashed line for a maturity of three years and the crossed line for 10 years. The initial debt level is $b_{-1}^N = b_{-2}^N = \dots = b_{-N}^N = 0$.

Figure 8: Optimal Portfolio under No Buyback - 'Lending Model'



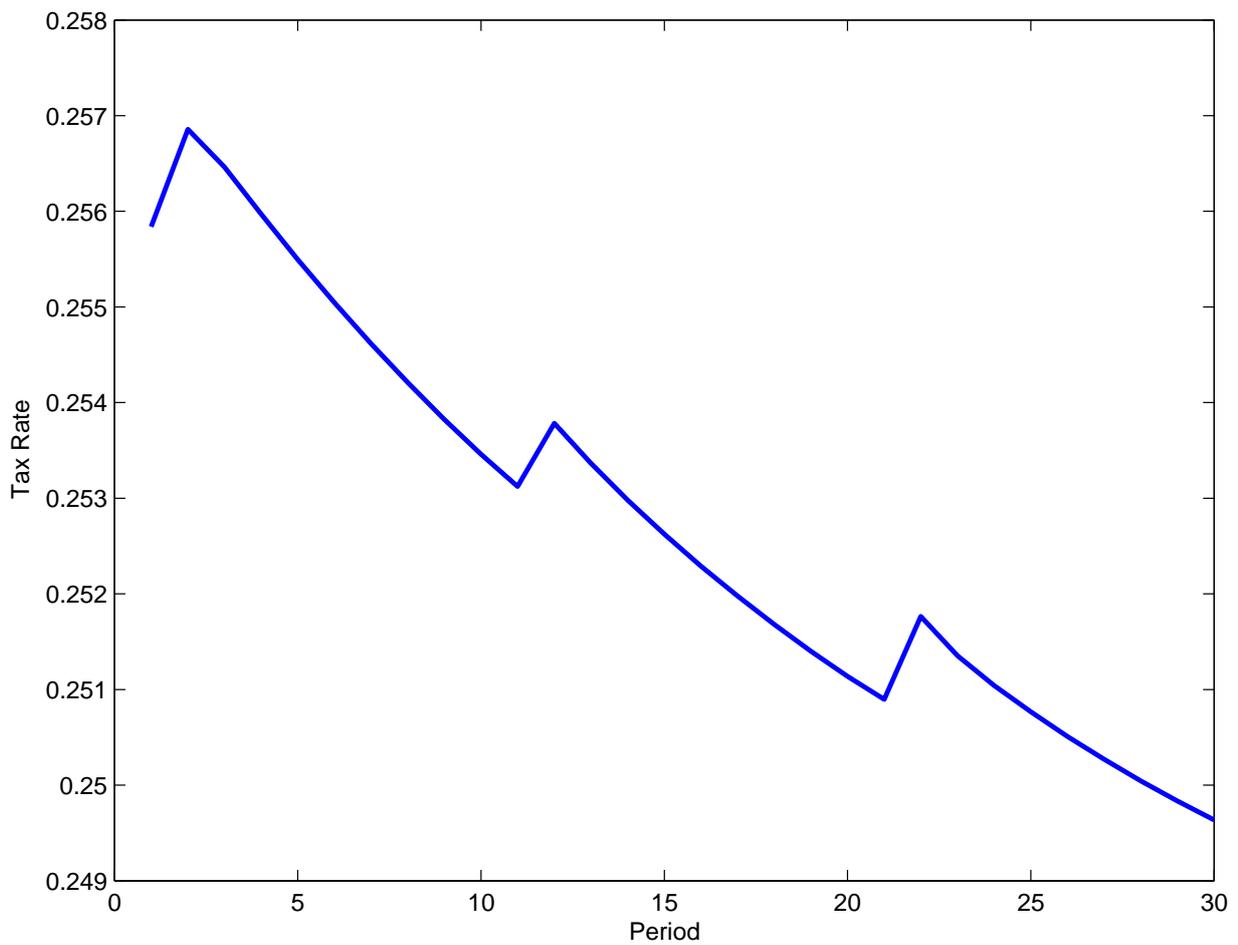
Notes: The Figure plots a sample path of the optimal portfolio under no buyback. The bounds are at 150% of (steady state) GDP. The solid line represents the short term bond. The dashed line the long maturity bond.

Figure 9: **Optimal Portfolio under No Buyback - 'No Lending' Model**



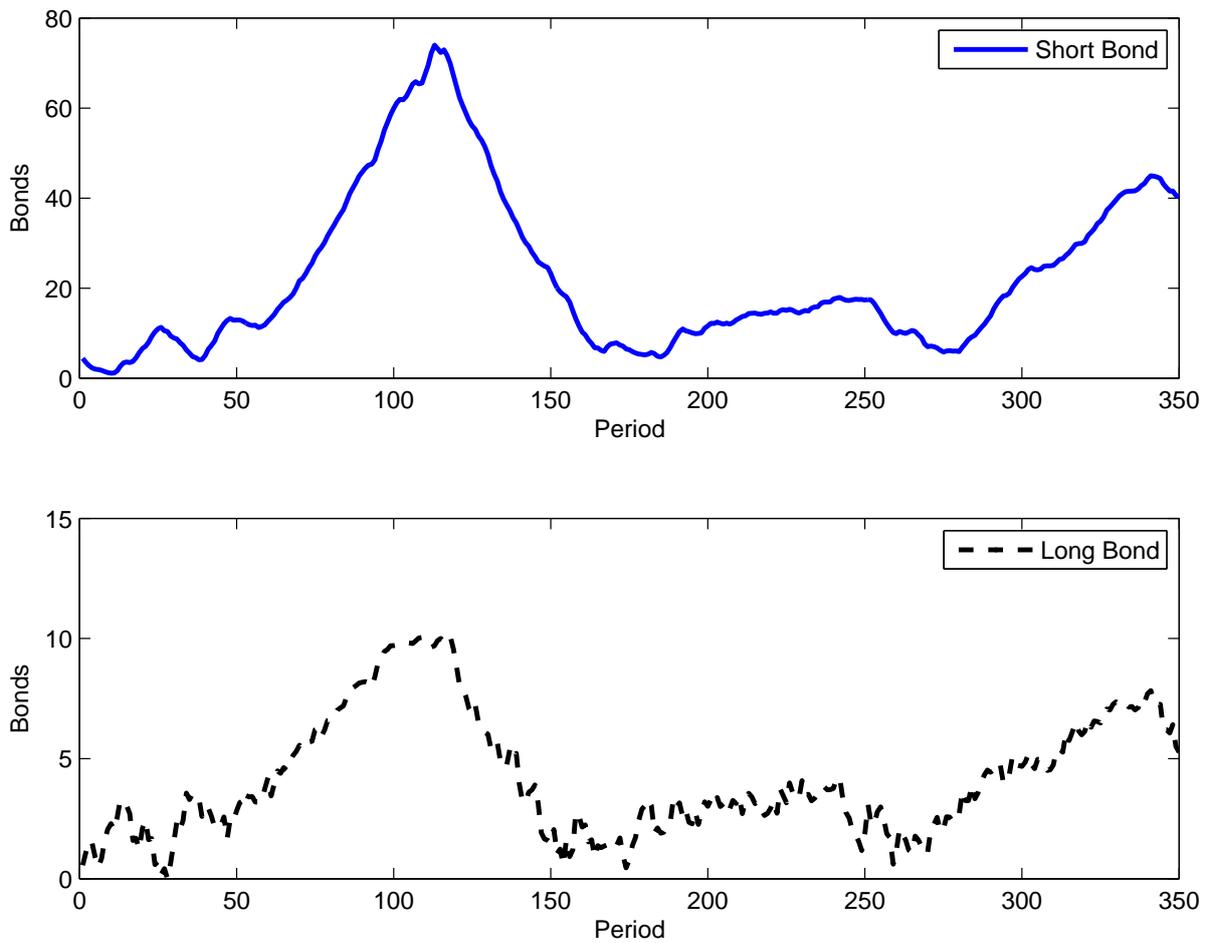
Notes: The Figure plots a sample path of the optimal portfolio under no buyback and 'No Lending'. The upper bounds are at 150% of (steady state) GDP. The solid line represents the short term bond. The dashed line the long maturity bond.

Figure 10: **Response of the Tax Schedule with Non-Zero Coupon Long Bonds**



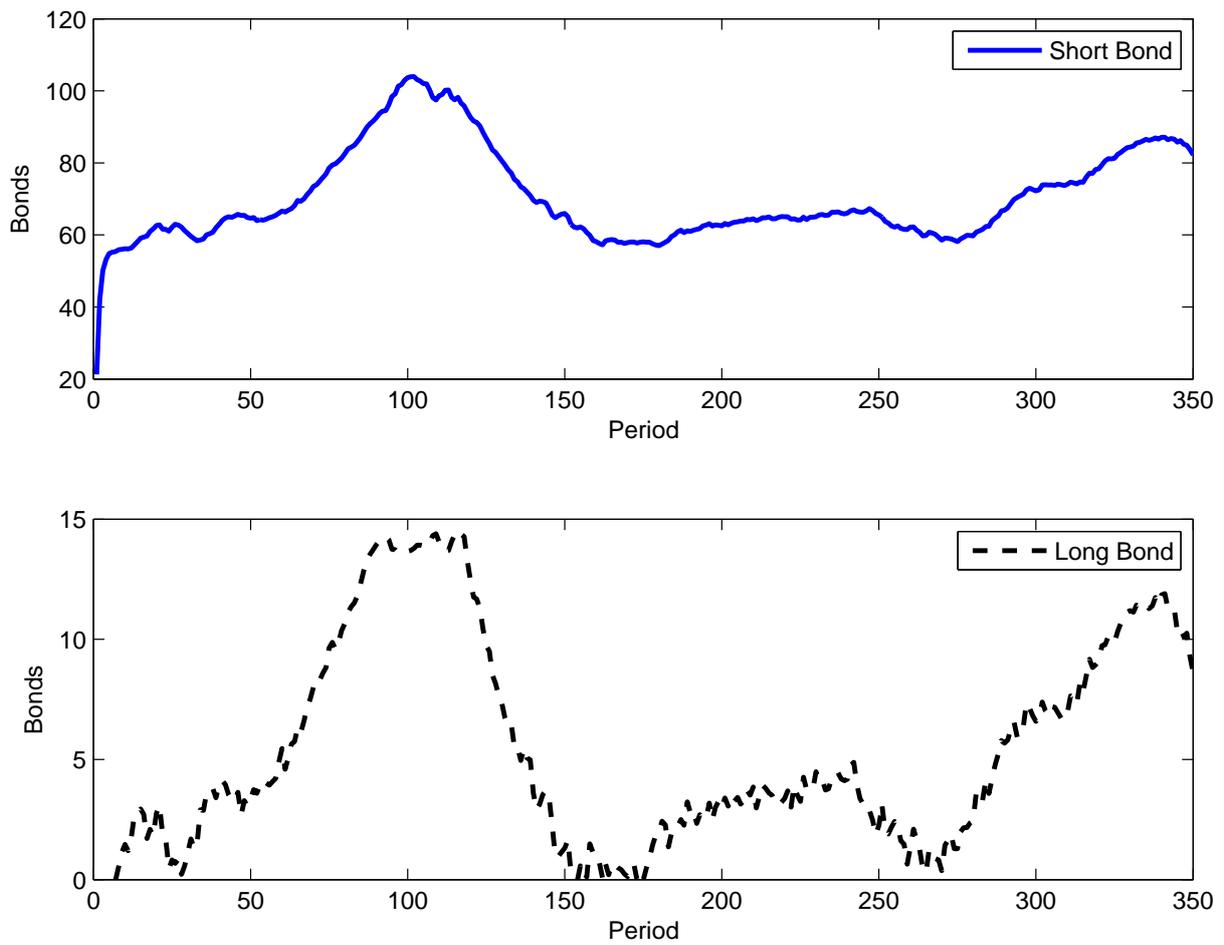
Notes: The Figure plots the response of the tax rate to a fiscal shock in a model where the government can issue debt in only a ten year non-zero coupon bond.

Figure 11: Optimal Portfolio with Non- Zero Coupon Bonds - 'No Lending'



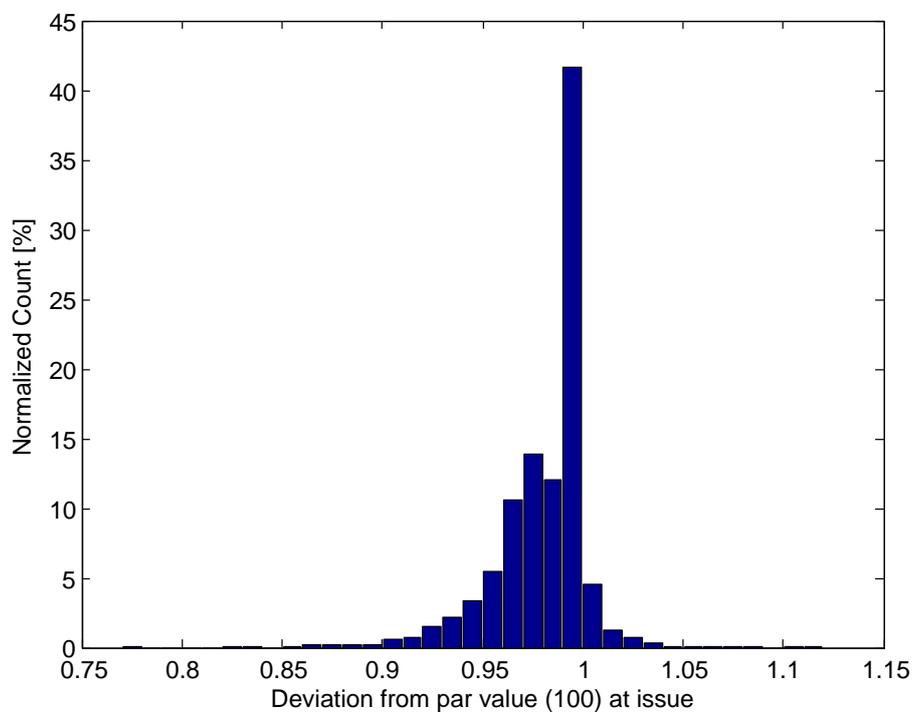
Notes: The Figure plots a sample path from no buyback model with non zero coupon bonds and 'No Lending'. The top panel illustrates the behavior of short maturity debt. The bottom panel plots the long term bond.

Figure 12: Optimal Portfolio with Buybacks After 5 Years



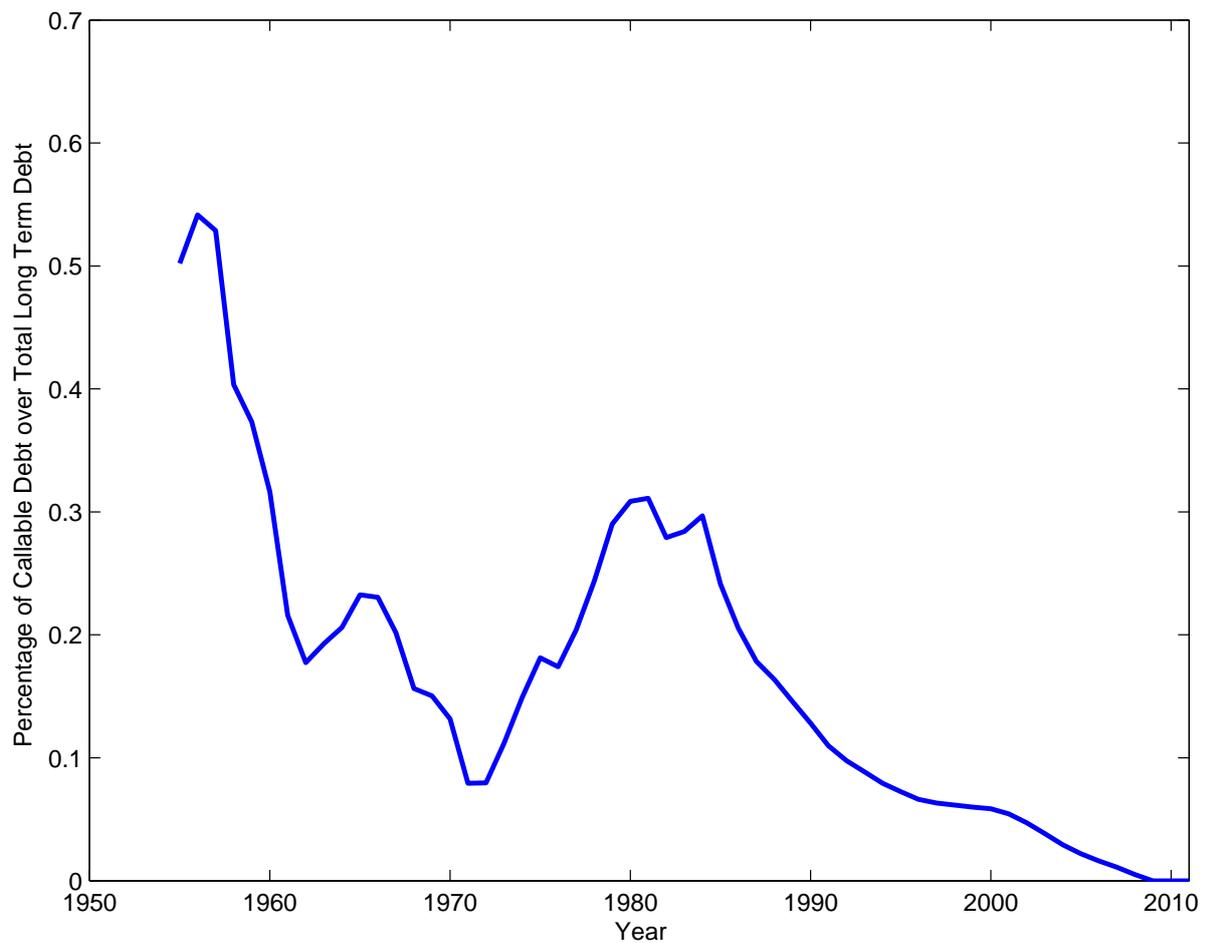
Notes: The Figure plots a sample path assuming that the government can buy ten year debt five years after its issuance. The top panel displays short term debt. The bottom panel shows long term debt.

Figure 13: Prices of Non Zero Coupon Bonds - Deviations from Par Value



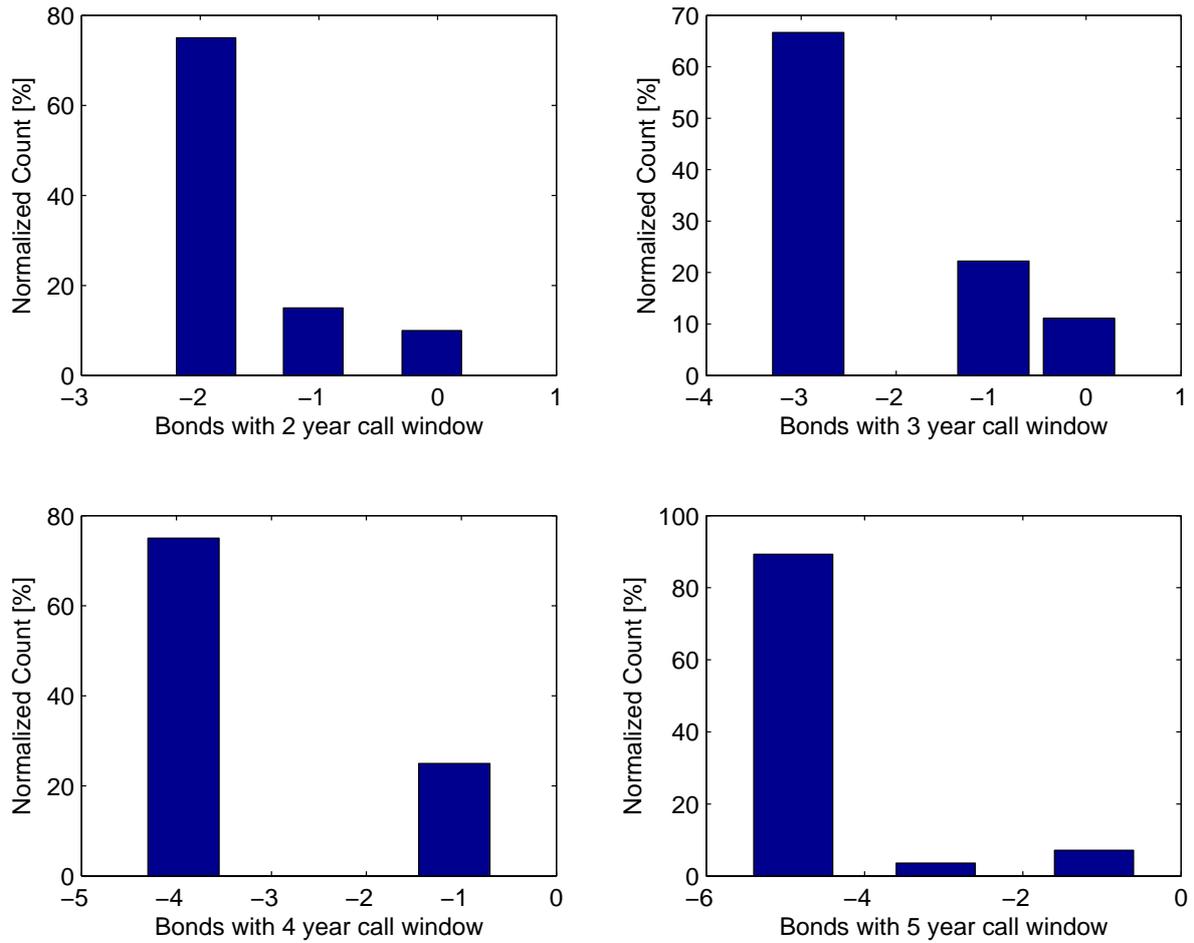
Notes: The Figure plots the distribution of bond prices of non zero coupon bonds issued since 1955 in the US. Par value is represented by one on the horizontal axis.

Figure 14: **Callable Bonds over Long Bonds in the US data**



Notes: The Figure plots the fraction of long term debt outstanding in the US which is callable. The data are for the years 1955-2011. The data source is the CRSP.

Figure 15: Callable Bonds over Long Bonds in the US data



Notes: The plots shows the timing of redemptions of callable debt in the US for all callable securities between 1955-2011. The top left shows the timing for bonds whose first call date is 2 years before maturity, the top right 3 years, and the bottom panels 4 and 5 years.

Government Debt Management: The Long and the Short of it*

Online Appendix

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A Online Appendix

A.1 Portfolio Reversals under Complete Markets

In the simulations of the standard buyback model with ‘lending’ and incomplete markets described in section 5.1 in the paper, it occurs that in some periods the prediction of saving in short bonds and getting in debt in long bonds is reversed. In particular this occurs between period 120 and 150 of our simulation in Figure 4. In the text we claimed that reversing the positions (issuing short term debt) is a feature of the complete market allocation in cases where the government has initial positive wealth. We derive here analytical results to support this claim.

Consider the (more general) case where g_t is represented as a serially correlated AR(1) process but where the innovation to spending can take only two values in a given period, i.e.

$$g_t = \mu + \rho g_{t-1} + \epsilon_t$$

and $\epsilon_t \in \{\epsilon^H, \epsilon^L\}$ (with some probabilities $p^H, (1-p^H)$). Notice that though g_t can take a continuum of values the uncertainty about $t+1$ from the point of view of information at t only takes two values. Therefore the government can complete the market through taking positions only in two maturities.¹

Following standard results (eg. Angeletos (2002), Buera and Nicolini (2006)) we can claim that under a complete market taxes, consumption, labor and the two maturities depend on the value of g_t only. Let us further assume (for the sake of deriving analytical results) that household utility is represented by the following function:

$$\log c_t - B \log l_t$$

where $l_t = T - x_t$ represents hours worked in the market.

In this case, it can be established that a complete market allocation features optimal taxes which are constant, so that from the household’s optimization we have:

$$c_t = K l_t$$

where $K = \frac{B}{1-\bar{\tau}}$ and $\bar{\tau}$ represents the (constant) tax rate in the economy. Combining with feasibility ($c_t + g_t = l_t$) we have:

$$l_t = \bar{K} g_t$$

for an appropriate constant $\bar{K} \equiv (1 - K)$. Moreover, the primary deficit satisfies:

$$g_t - \tau l_t = \bar{K} g_t.$$

To compute the complete market portfolios we follow the approach of Buera and Nicolini (2006). Let the present value of the deficit (given a history g^{t-1} and the realization of spending in t g_t) be given by:

$$(1) \quad z_t(g^{t-1}, g_t) \equiv E_t \left[\sum_{j=0}^{\infty} \frac{\beta^j u_{c,t+j}}{u_{c,t}} (g_{t+j} - \tau l_{t+j}) \right] | g^{t-1}, g_t.$$

¹See Shin (2007) for a similar example on escalating wars with two maturities and complete financial markets.

We then need to reproduce the variability in the z 's with the bond portfolio. After appropriate substitutions we have:

$$z_t(g^{t-1}, g_t) \equiv K \bar{K} g_t E_t \left(\sum_{j=0}^{\infty} \beta^j \frac{\bar{\bar{K}} g_{t+j}}{K \bar{K} g_{t+j}} \middle| g^{t-1}, g_t \right) = \bar{\bar{K}} g_t$$

for a constant $\bar{\bar{K}}$.

Notice that we know the following about this constant: if the government is initially in debt $\bar{\bar{K}} < 0$, and if initially the government has savings the opposite holds (i.e. $\bar{\bar{K}} > 0$). At zero initial debt is zero it holds that $\bar{\bar{K}} = 0$. Moreover, $z_t(g^{t-1}, g_t)$ can be written as

$$\begin{aligned} \bar{z}^H(g_{t-1}) &= \bar{\bar{K}}(\mu + \rho g_{t-1} + \varepsilon^H) \\ \bar{z}^L(g_{t-1}) &= \bar{\bar{K}}(\mu + \rho g_{t-1} + \varepsilon^L) \end{aligned}$$

Assume we have a long bond of maturity N and a one-period bond. We need to find portfolios such that

$$(2) \quad -b_{t-1}^1(g^{t-1}) - \bar{p}^{N-1}(g_t) b_{t-1}^N(g^{t-1}) = \bar{z}^i(g_{t-1}) \quad \text{for } i = H, L \quad \forall t$$

where we are imposing that the bond price at t depends only on g_t . If we take a large N so that we have that $E_t(c_{t+N}^{-1})$ is arbitrarily close to the unconditional mean $E(c_t^{-1})$ the bond price can be written approximately as

$$\bar{p}^N(g_t) = \beta^N g_t \tilde{K}$$

Therefore, for large N

$$\begin{aligned} -b_{t-1}^1(g^{t-1}) - \beta^{N-1} g_t \tilde{K} b_{t-1}^N(g^{t-1}) &= \bar{\bar{K}}(\mu + \rho g_{t-1} + \varepsilon^H) \\ -b_{t-1}^1(g^{t-1}) - \beta^{N-1} g_t \tilde{K} b_{t-1}^N(g^{t-1}) &= \bar{\bar{K}}(\mu + \rho g_{t-1} + \varepsilon^L) \end{aligned}$$

Subtracting one equation from the other we get:

$$\beta^{N-1} b_{t-1}^N(g^{t-1}) = -\frac{\bar{\bar{K}}}{\tilde{K}}$$

so that $\beta^N b_{t-1}^N(g^{t-1})$ is a constant. Note that if $\bar{\bar{K}}$ is negative (ie the government is initially in debt) then $b^N > 0$ as in Angeletos (2002) and Buera and Nicolini (2006), but if the government has initial savings the government buy a lot of long private bonds.

Moreover we can find short bonds from

$$\begin{aligned} -b_{t-1}^1(g^{t-1}) &= \bar{\bar{K}}(\mu + \rho g_{t-1} + \varepsilon^H) + \beta^{N-1} g_t \tilde{K} b_{t-1}^N(g^{t-1}) \\ &= \bar{\bar{K}} g_t - g_t \tilde{K} \frac{\bar{\bar{K}}}{\tilde{K}} = 0 \end{aligned}$$

So short bonds should be zero.

These derivations support our claim that long run savings are a feature of complete markets if the government owns negative debt. As discussed in text we have confirmed that with our baseline preferences we can get portfolios with reversed positions under complete markets if government debt is very negative.

A.2 Complete Markets and No Buyback

We describe in this section the debt management strategy under a complete financial market assuming no buyback.

As in Chari and Kehoe (1999) we formulate the planner's program as a maximization of the household's utility subject to the following implementability constraint at date 0:

$$E_0 \sum_{t=0}^{\infty} \beta^t u_{c,t} [(u_{c,t} - v_{x,t})(c_t + g_t) - u_{c,t} g_t] = E_0 \sum_{i \in \{1, N\}}^N \sum_{t=0}^{i-1} \beta^t u_{c,t} b_{t-i}^i$$

where for simplicity we assume that the government may issued debt in one year and N year maturities. Note that the above condition is equal to the equation (27) derived in Section 6 in paper.

In order to characterize the optimum we use the Lagrangian approach. The optimal allocation derives as a solution to the following equation:

$$(3) \quad \mathcal{L}_{CM} = E_0 \left\{ \sum_{t=0}^{\infty} [\beta^t u(c_t) + v(T - c_t - g_t)] - \Lambda \left[\sum_{t=0}^{\infty} \beta^t u_{c,t} ((u_{c,t} - v_{x,t})(c_t + g_t) - u_{c,t} g_t) - \sum_{i \in \{1, N\}}^N \sum_{t=0}^{i-1} \beta^t u_{c,t} b_{t-i}^i \right] \right\}$$

where Λ represents the constant multiplier on the implementability constraint of the government.

The first order conditions for the optimum are given by:

$$(4) \quad \Gamma_t - \Lambda[u_{cc,t} c_t + u_{c,t} - v_{x,t} + v_{xx,t}(c_t + g_t)] = 0 \quad \text{for } t > N - 1$$

$$(5) \quad \Gamma_t - \Lambda[u_{cc,t} c_t + u_{c,t} - v_{x,t} + v_{xx,t}(c_t + g_t)] + \Lambda u_{cc,0} \sum_{i \in \{1, N\}} b_{t-i}^i = 0 \quad \text{for } t \leq N - 1$$

where $\Gamma_t = u_{c,t} - v_x(T - x_t)$.

A.2.1 Optimal Debt Management

Let $g^{t-1} = (g_0, g_1, \dots, g_{t-1})$ be the history of government spending shocks up to date $t - 1$. We define as in Angeletos (2002) and Buera and Nicolini (2004) and Faraglia et al. (2010) z as the present discounted value of the government surplus contingent on g^{t-1} and the current realization of spending g_t .

$$(6) \quad z_t(g^{t-1}, g_t) = E_t \sum_{i=0}^{\infty} \frac{\beta^i}{u_{c,t}} [(u_{c,t+i} - v_{x,t+i})(c_{t+i} + g_{t+i}) - g_{t+i} u_{c,t+i}]$$

Assuming that debt is not state contingent, that there are enough maturities to effectively com-

plete the markets and that the asset returns are not perfectly correlated, one of the key results of Buera and Nicolini (2004) is that z is the only variable (along with the asset prices) that is needed to pin down the optimal issuance of the government. In this case we can follow the same approach.

Let's recall the simple example of Section 2 of Faraglia et al. (2010). We assume for simplicity that government expenditure follows a two step Markov process taking values $g_H > g_L$ with probabilities, π_{HH} and π_{LL} , of remaining in the same state. We need only two risk free bonds to complete the markets. We assume that the government has access only to a one period bond, b^1 , and a bond of maturity N , b^N . If there is no initial debt we have already stated that all the time t variables will depend only on g_t and in particular $z_t(g^{t-1}, g_i) = z_i$ and $p^{N-1}(g^{t-1}, g_i) = p_i^{N-1}$ for $i = H, L$ and for all t . If we further assume that $g_0 = g_H$ then $z_H = 0 < z_L$.

The optimal portfolio must be such that the outstanding value of the government's liability is equal to 0 whenever government spending is high and equal to z_L when spending is low. If we first assume that the government has to buy back all the outstanding debt in every period, the optimal portfolio needs to satisfy:

$$(7) \quad b_{t-1}^1(g^{t-1}) + p_i^{N-1}(g^{t-1}) b_{t-1}^N(g^{t-1}) = z_i \quad \text{for } i = H, L \quad \forall t$$

Assuming $p_H^{N-1} \neq p_L^{N-1}$ the unique solution to the problem is given by:

$$(8) \quad \begin{pmatrix} 1 & p_H^{N-1} \\ 1 & p_L^{N-1} \end{pmatrix} \begin{pmatrix} b_{t-1}^1(g^{t-1}) \\ b_{t-1}^N(g^{t-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ z_L \end{pmatrix}.$$

yielding

$$(9) \quad \begin{pmatrix} b_{t-1}^1(g^{t-1}) \\ b_{t-1}^N(g^{t-1}) \end{pmatrix} = \begin{pmatrix} \frac{p_H^{N-1} z_L}{p_H^{N-1} - p_L^{N-1}} \\ \frac{-z_L}{p_H^{N-1} - p_L^{N-1}} \end{pmatrix} = \begin{pmatrix} B^1 \\ B^N \end{pmatrix}$$

The main characteristics of the solution are that firstly the issuance of each maturity is constant over the business cycle and secondly that if a standard utility function is assumed such that $p_H^{N-1} < p_L^{N-1}$ then $B^1 < 0$ and $B^N > 0$.

Consider now the case where the government has to hold all the debt to maturity. The analogous of equation (7) assuming no buyback is:

$$(10) \quad b_{t-1}^1(g^{t-1}) + \sum_{j=1}^N p_i^{N-j}(g^{t-1}) b_{t-j}^N(g^{t-1}) = z_i \quad \text{for } i = H, L \quad \forall t$$

that states that in order to complete the markets the government will issue bonds ensuring that the total value of debt each period equals the discounted sum of surpluses z_i . Hence the total wealth of the consumer, including unmatured bonds, has to equal the discounted sum of future deficits. Given initial conditions b_{-1}^1 and b_{-i}^N for $i = 1, \dots, N - 1$ this equation describes the whole evolution of b_t^1 and b_t^N . To evaluate the implications of this equation we focus on the steady state.

In our two state example a generalization of (8) gives:

$$(11) \quad \begin{pmatrix} 1 & \sum_{i=1}^{N-1} p_H^i \\ 1 & \sum_{i=1}^{N-1} p_L^i \end{pmatrix} \begin{pmatrix} b_{t-1}^1(g^{t-1}) \\ b_{t-1}^N(g^{t-1}) \end{pmatrix} = \begin{pmatrix} 0 \\ z_L \end{pmatrix}.$$

yielding

$$(12) \quad \begin{pmatrix} b_{t-1}^1(g^{t-1}) \\ b_{t-1}^N(g^{t-1}) \end{pmatrix} = \begin{pmatrix} \frac{\sum_{i=1}^{N-1} p_H^{N-i} z_L}{\sum_{i=1}^{N-1} (p_H^{N-i} - p_L^{N-i})} \\ \frac{-z_L}{\sum_{i=1}^{N-1} (p_H^{N-i} - p_L^{N-i})} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_{ss}^1 \\ \mathbf{B}_{ss}^N \end{pmatrix}$$

For standard utility functions it holds that $p_H^i < p_L^i$ for all $i = 1, \dots, N$ so that, as before, $\mathbf{B}_{ss}^N > 0$, showing that in this economy with only expenditure shocks in steady state the government should issue long bonds in order to mop up the variations in z_t . The government has also to save in short bonds, $\mathbf{B}_{ss}^1 < 0$.

A.2.2 Relative Debt Positions

Let us now compare these steady state positions with the quantities given by (9). In the case of $N > 2$ we can expect $\left| \sum_{i=1}^{N-1} (p_H^{N-i} - p_L^{N-i}) \right| > |p_H^{N-1} - p_L^{N-1}|$ implying $\mathbf{B}_{ss}^N < B^N$. In other words, in steady state the government will issue lower amounts of long term bonds every period if it holds the bonds until maturity. This is not surprising, any long bond now constitutes debt for the following N periods, so that less debt needs to be issued every period.

In order to compare the debt positions when buyback is ruled out we shall focus on the ratio of the *value* of total long debt (*RVLD*):

$$RVLD^j \equiv \frac{p_j^{N-1} B^N}{\left(\sum_{i=1}^{N-1} p_j^i \right) \mathbf{B}_{ss}^N}$$

that compares the value of total government long term outstanding debt in the model with buyback (in the numerator) with the value of total long term outstanding debt in the model without buyback. Even though bonds are constant at steady state, this ratio depends on the realization $j = H, L$ due to the fact that prices change with the current realization j .

To gain some insight on likely values of this ratio, we assume that the process for government spending shocks g i.i.d. and that β close to 1. Under these assumptions we can show that

$$(13) \quad E(RVLD) \approx 1 - \frac{1}{N}$$

Therefore, according to (13), ruling out buyback may increase the value of total long debt held by the government, since $1 - \frac{1}{N} < 1$. Moreover, for the short term bond we have that:

$$\frac{B^1}{\mathbf{B}_{ss}^1} = RVLD^H$$

where the equality uses the first equations in (12) and in (9) and the definition of $RVLD^H$. Therefore, except for the discrepancy between $RVLD^H$ and $E(RVLD)$, which is likely to be small, we can claim

that for the case of i.i.d. expenditure shocks and β close to 1 we have that:

$$\frac{B^1}{\mathbf{B}_{ss}^1} \approx 1 - \frac{1}{N}.$$

Therefore ruling out buyback leads also to larger positions in short bonds as well. As N grows the difference becomes smaller.

A.3 The Ramsey Program under Non-Zero Coupon Bonds

We solve here the optimal policy problem assuming that the government issues debt in one year and N year bonds (see section 7.1 of the paper). As discussed for the non-zero coupon N year bond the debt limits are:

$$b_t^N \in \left[\frac{\underline{M}_N}{\sum_{j=1}^N \beta^j + \kappa_N \sum_{j=1}^N \sum_{i=1}^j \beta^i}, \frac{\overline{M}_N}{\sum_{j=1}^N \beta^j + \kappa_N \sum_{j=1}^N \sum_{k=1}^j \beta^k} \right] \equiv [\widetilde{M}_N, \overline{\widetilde{M}}_N]$$

Letting $[\widetilde{M}_1, \overline{\widetilde{M}}_1] \equiv [\frac{M_1}{\beta}, \frac{\overline{M}_1}{\beta}]$ be the analogous constraint set for one year debt we represent the planning problem is given by ²:

$$\begin{aligned} \mathcal{L} = E_0 \sum \beta^t & \left\{ u(c_t) + v(T - c_t - g_t) - \lambda_t \left[b_t^1 \beta u_{c,t+1} + b_t^N (\beta^N u_{c,t+N} + \sum_{j=1}^N u_{c,t+j} \kappa_N) \right. \right. \\ & \left. \left. - b_{t-1}^1 u_{c,t} - b_{t-N}^N u_{c,t} - \kappa_N \sum_{j=1}^N b_{t-j}^i u_{c,t} - g_t u_{c,t} + (u_{c,t} - v_{x,t})(g_t + c_t) \right] \right. \\ & \left. + \sum_{i \in \{1, N\}} \xi_{U,t}^i (\overline{\widetilde{M}}_i - b_t^i) + \sum_{i \in \{1, N\}} \xi_{L,t}^i (b_t^i - \widetilde{M}_i) \right\} \end{aligned}$$

The first order condition for consumption is :

$$\begin{aligned} & u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t}) \\ & + u_{cc,t} \kappa_N \sum_{j=1}^N (\lambda_{t-j} - \lambda_t) b_{t-j}^i + u_{cc,t} \sum_{i \in \{1, N\}} (\lambda_{t-i} - \lambda_t) b_{t-i}^i = 0 \end{aligned}$$

and off corners the analogous conditions for b_t^1 and b_t^N are:

$$(14) \quad \lambda_t E_t(u_{c,t+1}) = E_t(\lambda_{t+1} u_{c,t+1})$$

$$(15) \quad \lambda_t E_t(\kappa_N \sum_{j=1}^N \beta^j u_{c,t+j} + \beta^N u_{c,t+N}) = E_t(\kappa_N \sum_{j=1}^N \beta^j u_{c,t+j} \lambda_{t+j} + \beta^N u_{c,t+N} \lambda_{t+N})$$

A.4 Ramsey Program in the Case of Buybacks m Years after Issuance

We derive the Ramsey program for the simulations shown in Section 7.2. As explained in the text we assume that the government buys back its outstanding liabilities m years after the issuance of a

²In the US coupon payments are usually six monthly. However, since our model's horizon is one year we model one year debt as zero coupon. This is obviously innocuous.

bond. We further assume that long term bonds pay zero coupons. Under these assumptions the per period budget constraint is given by:

$$\sum_{i \in \{1, N\}} p_t^i b_t^i = b_{t-1}^1 + p_t^{N-m} b_{t-m}^N + g_t - \tau_t(T - x_t)$$

where for simplicity we impose that the government trades a short maturity of one year.

Since we assume here that when a bond is issued it stays in the market for m years the appropriate formulation of the ad hoc debt constraints is for the N year bond:

$$b_t^N \in \left[\frac{\underline{M}_N}{\sum_{j=1}^m \beta^j}, \frac{\overline{M}_N}{\sum_{j=1}^m \beta^j} \right] \equiv [\widetilde{M}_N, \widetilde{\overline{M}}_N]$$

Letting $[\widetilde{M}_1, \widetilde{\overline{M}}_1]$ be the analogous constraint set for one year debt and substituting the equilibrium expressions for the tax rate and the bond prices we represent the planning problem as follows:

$$\begin{aligned} \mathcal{L} = E_0 \sum_t \beta^t \left\{ u(c_t) + v(T - c_t - g_t) - \lambda_t \left[\sum_{i \in \{1, N\}} b_t^i \beta^i u_{c, t+i} - b_{t-1}^i u_{c, t} - b_{t-m}^N \beta^{N-m} u_{c, t+N-m} \right. \right. \\ \left. \left. - g_t u_{c, t} + (u_{c, t} - v_{x, t})(g_t + c_t) \right] \right. \\ \left. + \sum_{i \in \{1, N\}} \xi_{U, t}^i (\widetilde{M}_i - b_t^i) + \sum_{i \in \{1, N\}} \xi_{L, t}^i (b_t^i - \widetilde{M}_i) \right\} \end{aligned}$$

The first order conditions for the optimum are given by:

$$\begin{aligned} u_{c, t} - v_{x, t} + \lambda_t (u_{cc, t} c_t + u_{c, t} + v_{xx, t} (c_t + g_t) - v_{x, t} + u_{cc, t} [(\lambda_{t-1} - \lambda_t) b_{t-1}^1 + (\lambda_{t-N} - \lambda_{t-N+m}) b_{t-N}^N \\ \beta E_t (u_{c, t+1} \lambda_{t+1} - u_{c, t+1} \lambda_t) + \xi_{L, t}^1 - \xi_{U, t}^1 = 0 \\ \beta^N E_t (u_{c, t+N} \lambda_{t+m} - u_{c, t+N} \lambda_t) + \xi_{L, t}^N - \xi_{U, t}^N = 0 \end{aligned}$$

A.5 Robustness

In this section we test the robustness of our results to different parameterizations of the model.

A.5.1 Different Maturity Structures

Our results could potentially be affected if we assume a different maturity structure for government debt than we have thus far. Note that in the buyback model of Section 5, we illustrated that the hedging value of long bonds induces the government to want to borrow (mostly) long term. Moreover, under the case of no buyback we showed that the tax smoothing objective compels the government to use short term debt, when fiscal policy under long bonds follows an N period cycle. It seems that shortening the duration of the long maturity could potentially impact the optimal debt management strategy. We investigate in this section whether it is the case.

To show that our results are robust to a different maturity structure we assume that the government has access to a five rather than a ten year bond as the long maturity. Note that in this case

the N cycle becomes shorter, thus impling less tax distortions associated with long maturity in the no buy back model.

Figure 1 shows that the implications for optimal debt management are largely unaffected by this change in the maturity. Under no buy back the optimal policy is once again to utilize both short term and long term debt to finance spending.³

[Figure 1 About Here]

B Numerical Appendix

This paper has proposed two modifications of the Parameterized Expectation Algorithm (PEA) of den Haan and marcet (1990) in order to solve the optimal fiscal policy problem with two maturities:

1. Using the ‘Forward States’ method we have been able to deal with the indeterminacy of the optimal portfolio, arising in the standard stochastic PEA approach;
2. Using ‘Condensed PEA’ we have been able to deal with the large number of state variables in the model. In a numerical paper on “Optimal Fiscal Policy Problems under Complete and Incomplete Financial Markets: a Numerical Toolkit” (FMOS (2014 (a))) we discuss in detail the implementation of these methodologies applying them to the optimal portfolio choice problem (this paper) but also (for the case of the ‘Condensed PEA’) to an optimal policy problem with one long term bond (e.g. FMOS (2014 (b))). We refer the reader to that paper for further details. Here we briefly outline some of the features of the numerical solution and comment on alternative approaches that the literature proposed to solve for optimal portfolios and to deal with large scale applications.

B.1 Practical Features of the Solution

B.1.1 Approximating Polynomials

In FMOS (2014 (a)) we illustrate that it is sufficient to assume that Φ^i and Ψ^i are linear in the state variables. This is due to two properties of the incomplete market model. First the fact that debt is close to a random walk means that the policy rule is linear and close to the 45 degree line.⁴ Second, nonlinearities which may be important when the economy is close to a debt constraint are not significant here since the constraints are very loose.

The only case where a nonlinearity emerges and needs to be explicitly treated in the approximations is when there is a ‘No Lending’ constraint. In this case we include to the approximation slines of the following form:

$$I_t^{L,i} = \begin{cases} b_t^i - \xi & \text{if } b_t^i \leq \xi \\ 0 & \text{otherwise} \end{cases}$$

such that the slope of debt in Φ^i and Ψ^i change when b_t^i is less than some fixed level ξ . In FMOS (2014 (a)) we show that the inclusion of $I_t^{L,i}$ is preferable to the inclusion of higher order terms in the approximating polynomials.

³For brevity we plot the case of the ‘No Lending’ model under no buyback. Our results are similar if we consider the case of the ‘Lending’ model.

⁴Under no buyback linearity emerges with the inclusion of the N -th lag of the bond quantity and the multiplier.

B.1.2 Dealing with the Debt Constraints.

We illustrated in text that when the optimal debt level violates one of the bounds the solution to the model does not satisfy an Euler equation. In this case the computer program is set to find a consumption level and a value for the multiplier in t so that the constraint of the government and the first order condition with respect to c_t hold. In the context of the single bond economies this can be easily programmed. Suppose that ignoring the debt limits we obtain, solving the non-linear system, $b_t^N > \frac{\overline{M}_N}{\beta^N}$. Then we can claim that the constrained optimum is one where $b_t^N = \frac{\overline{M}_N}{\beta^N}$.

When we have with two or more maturities in the model it is not as easy to deal with the constraints. The reason is that when a constraint is binding and the debt level of one of the bonds needs to be adjusted the researcher has to take into account the effect of this adjustment on the other bond quantity. Suppose that $b_t^N > \frac{\overline{M}_N}{\beta^N}$ but $b_t^1 < \frac{\overline{M}_1}{\beta^1}$ in an ‘unconstrained’ optimum. Then when we force $b_t^N = \frac{\overline{M}_N}{\beta^N}$, we have to verify that it still holds that b_t^1 does not violate the bound. If it does, we have to impose the bound on the one year bond and so on.

Given this remark it is clear that introducing ad hoc debt constraints in a model of multiple maturities can be messy in terms of representing the planner’s program on the computer. To make our numerical codes simpler and to avoid having to consider separately the numerous cases involved, we utilize an adjustment cost function of the following form:

$$C(b_t^i) = \begin{cases} \phi_1(b_t^i - \frac{\overline{M}_i}{\beta^i}) + \log(1 + \phi_1(b_t^i - \frac{\overline{M}_i}{\beta^i})) & \text{if } b_t^1 \geq \frac{\overline{M}_i}{\beta^i} \\ \phi_1(\frac{M_i}{\beta^i} - b_t^i) - \log(1 + \phi_1(\frac{M_i}{\beta^i} - b_t^i)) & \text{if } b_t^1 \geq \frac{\overline{M}_i}{\beta^i} \\ 0 & \text{otherwise} \end{cases}$$

for $i = 1, N$ and where ϕ_1 is a parameter which governs the penalty from deviating from the debt levels $\frac{M_i}{\beta^i}$ and $\frac{\overline{M}_i}{\beta^i}$. We choose a value of ϕ_1 equal to unity. Moreover, note that the function $C(b_t^i)$ is continuous at the bounds.

B.2 Alternative Approaches

In this paragraph we place our methodology in the context of the literature, describing alternative approaches that we could have followed and the advantages of our approach over them.

First, note that methods of approximating the state space X with a finite number of nodes and solving the planner’s program via value function iteration (essentially applying the techniques of Fahri (2010), or Marcet and Marimon (2012)) do not raise an indeterminacy problem. However, with such a large state space we feel that inevitably value functions run into curse of dimensionality problems. In contrast our approach which relies on simulations (what Judd et al. (2011) refer to as one node Monte Carlo methods) enables us to consider only a relevant part of the state space that is visited in the simulations.

It is worth noting that value function iterations have been recently applied to large scale problems using techniques which choose efficiently the approximation points, and therefore reduce computational time. For example Lustig et al. (2008) solve for a model with multiple assets under the Ramsey outcome using the Smolyak polynomials of Kubler and Krueger (2004). Moreover, Judd et al. (2012) have proposed an approach that relies on simulations to formulate an approximation of the value function around the ergodic set of state variables. We acknowledge that these techniques

make solving our problem with the value function more tractable, however, in the case of very long maturities we would still have to recover the policy rules from a value function of a large number of state variables.⁵ In contrast our ‘Condensed PEA’ enables us to effectively reduce the number of states in the approximating polynomials by summarizing the predictive power of (X_t^{out}) via linear combinations. Value functions on the other hand require the entire state space to formulate the policy problem.

Within the class of simulation based PEA there is an alternative, more conventional, route we could follow. As in Marcet and Singleton (1999) we could choose to approximate the term $E_t(u_{c,t+i}\lambda_{t+1})b_t^i$ by a function $H^i(X_t, \delta^i)$, and the term $E_t(u_{c,t+i})\lambda_t$ by a polynomial $G^i(X_t, \gamma^i)$. Then, theoretically, the indeterminacy problem is no longer present: we could recover the bond holdings for different maturities using the following equation:

$$(16) \quad b_t^i = \frac{E_t(u_{c,t+i}\lambda_{t+1})b_t^i}{E_t(u_{c,t+i})\lambda_t} = \frac{H^i(X_t, \delta^i)}{G^i(X_t, \gamma^i)}$$

Note that this approach may lead to large gains in computation time, especially when the number of assets increases, because the researcher doesn’t have to deal with the task of finding numerical solutions to a complicated system of nonlinear equations as in (10) in the paper. If we used (16) we could get the paths of bonds with a long simulation and find solutions for consumption and the multiplier by solving a system of 2 equations for two unknowns. In contrast in (10) in the paper there are 4 equations for 4 unknowns, and if we had allowed for all maturities between 1 and N there would be $N + 2$ equations. Nonlinear systems involve costly computations of the Jacobian and the Hessian matrix and require good initial conditions to converge.⁶

While this different route seems to confer a computational advantage, we find that in practice it doesn’t work. When we parameterize the bond policy functions as in 16 the algorithm diverges or circles indefinitely. We suspect that this may partly be specific to solving a portfolio choice problem for the government under incomplete markets. Implementing the ‘Forward States’ method has been more successful.

Judd et al (2011 (b)) propose the idea of pre-computation of integrals. This methodology consists of making use of the equilibrium policy functions in the integrand terms of expectations to solve a value function or an Euler equation. This approach is used to remove uncertainty from the model, by integrating out future shocks to the state vector. Effectively by computing expectations just once, they save up on the time costs of numerical integration at each iteration of the model. This idea is similar to our Forward States modification to the PEA. However, our purpose is not to remove uncertainty from the model, which in the simulation based PEA is present in each iteration of the model, but rather to make use of the future state vector to resolve the indeterminacy.⁷

Finally, note that another approach that could be used in order to deal with the optimal portfolio choice, is the perturbation methods of Devereux and Sutherland (2011) applied to open economy models. There are two main reasons for why we are unhappy with using local methods in our context.

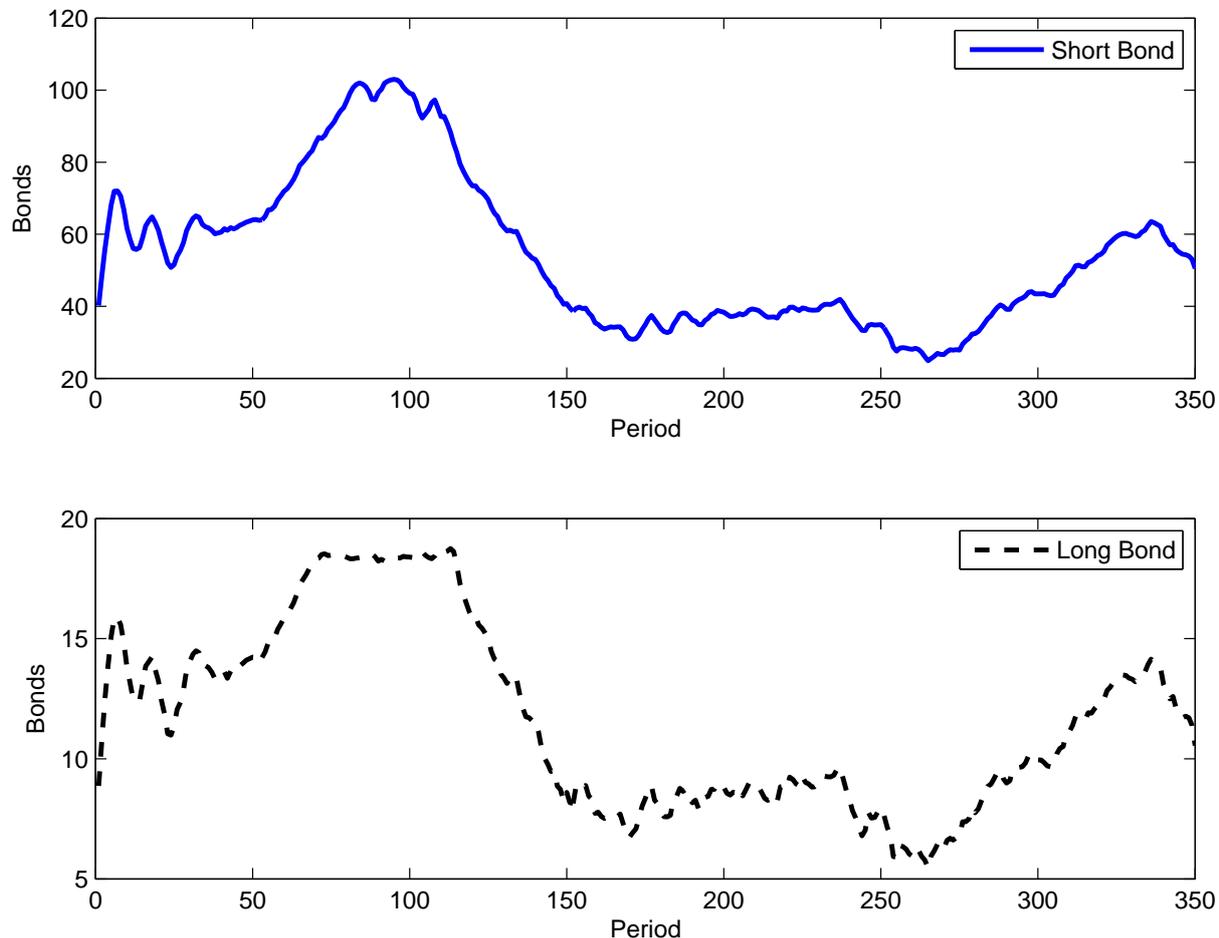
⁵In Lustig et al. (2008) the longest traded maturity is seven years, whereas our method offers the possibility to consider very long maturities.

⁶See Judd et al (2011) for a discussion.

⁷Judd et al (2011) apply their precomputation integrals to an open economy model, with twenty countries under complete markets. When they solve for country asset holdings they use Marcet and Singleton’s (1999) approach described previously. Therefore our application is different from theirs.

First, because of the random walk property of the multiplier λ_t discussed earlier, no single point can approximate the decision rules accurately in our context. Second, using local methods would preclude us from introducing limits to borrowing and lending for the government and these bounds are needed in order for markets to be incomplete.

Figure 1: **Optimal Portfolios under Five Year Long Maturity**



Notes: The Figure plots a sample path when the long maturity is five years. The solid line represents short term bonds and the dashed line long maturity bonds. The graphs correspond to the no buyback model under 'No Lending'.